

## MONEY AND THE GAINS FROM TRADE\*

BY ALEKSANDER BERENTSEN AND GUILLAUME ROCHETEAU<sup>1</sup>

*Economics Department (WWZ), University of Basel, Switzerland;  
School of Economics, Australian National University, Australia*

This article studies the role of money in environments where in each meeting there is a double coincidence of real wants. Traders who meet at random finance their purchases through current production, the sale of divisible money or both. It is shown that in the absence of valued money if traders have asymmetric tastes for each other's good, they produce and exchange socially inefficient quantities. With valued money, however, traders exchange efficient quantities if the asymmetry of tastes is not too large. It is shown that the gains from trade in the monetary economy are strictly greater than those in the corresponding barter economy, that the Friedman rule holds, and that the allocation of resources in the monetary economy converges to the allocation in the barter economy as the growth rate of the money supply is increased.

The need for trades to be “balanced” (satisfy the *quid pro quo*) at each exchange . . . conflicts with the potential for exploiting all the gains from trade. The role of money is to attenuate this conflict.

Ostroy and Starr (1990, p. 26)

### 1. INTRODUCTION

Models of bilateral trade have proven to be extremely helpful in capturing the nature of money.<sup>2</sup> In these models money is useful because it resolves the conflict between the need for trades to satisfy the *quid pro quo* requirement and the

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<sup>2</sup> The first models, developed by Ostroy (1973), Starr (1972), and Ostroy and Starr (1974), focused on the exchange process when bilateral trades are subject to a *quid pro quo* restriction and when prices are given and assumed to clear markets. In a second wave of models, Kiyotaki and Wright (1989, 1991, 1993) introduced random matching to represent the time-consuming trading process, to see what frictions can make monetary exchange an equilibrium, and to determine endogenously which objects serve as media of exchange.

potential for exploiting all the *gains from trade*.<sup>3</sup> A case in point is the random-matching framework pioneered by Kiyotaki and Wright (1991, 1993), which emphasizes the problem of *double coincidence of real wants*. In this framework, money improves welfare by increasing the *frequency* of trades because, with money, production and exchange can take place in single- as well as in double-coincidence meetings.

Recently, Engineer and Shi (1998), in a random-matching model with indivisible goods and indivisible money, have shown that money does not only allow agents to trade in single-coincidence meetings, but it sometimes also allows agents to attain higher gains from trade in double-coincidence meetings when they have asymmetric demands for each other's goods. Unfortunately, their formalization strategy has yielded ambiguous results, which has left us with the wrong impression that the *asymmetric-demand problem*, and consequently the benefit of money in their model, is less general than the *double-coincidence problem* emphasized in the random-matching literature.<sup>4</sup>

In this article, we study the role of money in a double-coincidence-of-wants environment (i.e., an environment where agents trade in all matches, even without money), and we show why the asymmetric demand problem encompasses the double-coincidence problem. For this purpose we consider a random-matching model with perfectly divisible money and divisible goods, where agents, endowed with money and production opportunities, have the choice of financing their consumption through current production, the sale of money, or both. The *terms of trade*, i.e., the quantities of goods that are produced, exchanged, and consumed in bilateral meetings, are determined through bilateral bargaining.

The key result of our model is that in a monetary economy the gains from trade are *strictly larger* than the gains from trade in the corresponding barter economy in *all* asymmetric matches. Thus, money indeed resolves the conflict between the quid pro quo requirement and the potential for exploiting all the gains from trade. The extent to which money mitigates this conflict depends on the real value of money. The higher the real value of money, the greater are the realized gains from trade in asymmetric meetings and the larger is the set of meetings where the traders exploit all the gains from trade. Consequently, by decreasing the real value of money, an increase in the inflation rate unambiguously deteriorates the allocation of resources. As the inflation rate increases, the economy moves gradually toward the allocation of the barter economy. Interestingly, the optimum quantity of money obeys the Friedman rule. Under this rule, the real value of money attains its maximum, and in each match all the gains from trade are realized.

<sup>3</sup> Quid pro quo requires that in any trade the value of the goods delivered by a trader must equal the value of the goods he receives. The origin of the quid pro quo restriction is that trades must be based on mutual agreements: a trade is neither a theft nor a gift.

<sup>4</sup> Engineer and Shi (1998) consider a random-matching model with indivisible goods and indivisible money, where the matched traders have asymmetric demands for each other's goods. To endogenize terms of trade, they introduce divisible service side payments over which agents can bargain. A crucial characteristic of these side payments is that the marginal cost of producing the services is always greater than the marginal utility of consuming them. Consequently, production of these side payments is a social waste. In Engineer and Shi's model, money is useful because it reduces the production of these inefficient side payments.

Money is useful because it modifies the terms of trade, i.e., the quantities of goods that are exchanged relative to the quantities the traders would exchange in the same situation in a barter economy. To capture this terms-of-trade effect, we deliberately exclude the frequency effect of money by considering an environment with *double coincidence of real wants*, where in each bilateral meeting each agent is a consumer of the other agent's production. In general, however, the bilateral meetings are *unbalanced* in the sense that one agent has stronger preferences for the good produced by his partner.<sup>5</sup> In such an environment, money cannot speed up the rate at which agents trade (the extensive margin); rather it modifies the quantities that agents produce and exchange (the intensive margin).

To explain how money affects the terms of trade, consider the bargaining between agents that have asymmetric preferences for each other's goods. Without money, they bargain over the quantities that are produced, consumed, and exchanged. The exchanged quantities simultaneously determine the total surplus of the match and how the traders split this surplus. Because agents cannot determine the size of the surplus and its split independently, and because each agent only cares about his own surplus, traders with asymmetric preferences attain a bargaining solution that does not exploit all the potential gains from trade.

With money, agents bargain over these quantities and over a monetary transfer. Money improves the terms of trade because it transforms a bargaining game without transferable utility into one with transferable utility. We therefore consider *utility transfer* as an important function of money.<sup>6</sup> In order to fulfill this function, (i) money must be divisible and (ii) it must be equally valued by all agents.<sup>7</sup> Consequently, a monetary transfer does not affect the size of the total surplus of a match, because changes in the payoffs cancel each other. This allows traders to separate the decisions of how much to produce and how to split the resulting total surplus. In contrast, real production is an *imperfect* device for transferring utility, because the marginal utility of the consumer and the marginal cost of the producer vary with the quantity produced and exchanged, and, in general, they do not coincide. Consequently, an exchange of real commodities does not affect the payoffs symmetrically, and therefore it affects the total surplus of the match.

We introduce asymmetric preferences by assuming that when two agents,  $i$  and  $j$ , meet, their preferences for each other's production good are represented by two random variables  $\varepsilon_i$  and  $\varepsilon_j$ , where  $\varepsilon_i$  measures how much agent  $i$  likes agent  $j$ 's

<sup>5</sup> Single-coincidence meetings, which are at the center in the random-matching literature, are meetings where agents have exceptionally strong asymmetric preferences: Only one agent likes to consume the good produced by his partner (see Figure 1).

<sup>6</sup> A bargaining game is a game with transferable utility if there is a device that allows a player to simultaneously decrease his own utility payoff and increase that of a partner by the same amount. A two-person bargaining game with transferable utility can be fully characterized by the disagreement payoffs of the two players and by the total transferable wealth available to the players (Myerson, 1991, pp. 384–85). Note that money transfers *indirect* utility, because money does not generate direct utility in our model: it is a claim on future consumption.

<sup>7</sup> Because agents have idiosyncratic preferences, money is the only good that is equally valued by all agents. This feature is related to one of Williamson and Wright (1994), who study a model where agents have private information about the goods they produce, and where the benefit of money arises because it is the only good that is of universally recognized quality.

good, and  $\varepsilon_j$  measures how much  $j$  likes  $i$ 's good. If  $\varepsilon_i > \varepsilon_j$ , agent  $i$  values agent  $j$ 's output more than  $j$  values  $i$ 's output when the same quantities are exchanged.

In a barter economy, efficiency is attained in symmetric matches only, i.e., if  $\varepsilon_i = \varepsilon_j$ . In asymmetric matches, however, agents produce and exchange inefficient quantities, and the degree of inefficiency increases in the degree of asymmetry. To see why bartering between asymmetric traders is inefficient, assume that  $\varepsilon_i > \varepsilon_j$ . From a social point of view, agent  $j$  should produce a larger quantity than agent  $i$ .<sup>8</sup> In a decentralized environment, because of the quid pro quo restriction, agent  $j$  will not agree to such an arrangement: in fact, we show that agent  $i$  produces more than agent  $j$ . This result illustrates in a drastic way the inefficiencies of bartering: the agent in a match who is more eager to consume produces more and consumes less than his partner. Consequently, in a barter economy, less-valued goods are overproduced and more highly valued goods are underproduced.

In contrast to the barter economy, a monetary economy also attains efficiency in asymmetric matches if the asymmetry, which we measure by some distance function  $\mathcal{D}(\varepsilon_i, \varepsilon_j)$ , is smaller than the real value of money holdings. The intuition behind this result is that, in contrast to the barter economy, agent  $j$  will agree to produce and exchange efficient quantities if agent  $i$  is able to compensate him by transferring claims to future consumption (money). If agent  $i$ 's constraint on money holding is not binding, then he is able to do so, and  $i$  and  $j$  produce and exchange efficient quantities. If not, the bargaining will result in inefficient quantities produced and exchanged.

Figure 1 illustrates, in the space of possible match types [for all  $(\varepsilon_i, \varepsilon_j)$ ], regions with efficient and inefficient trades: without money (Figure 1(a)), and with money for a certain inflation rate (Figure 1(b)). With money, the quantities exchanged are inefficient (efficient) for all  $\varepsilon = (\varepsilon_i, \varepsilon_j)$  in the shaded (nonshaded) region; see Figure 1(b). Without money, they are efficient on the 45-degree line and inefficient for all other  $\varepsilon$ ; see Figure 1(a). It is evident that the fraction of efficient trades is larger in the monetary economy. Moreover, the degree of inefficiency (welfare loss) in any of the matches in the grey area of the monetary economy is strictly smaller than the degree of inefficiency of the corresponding match in the barter economy. Note that in Figure 1(b) the shaded region of inefficient trades shrinks when the inflation rate decreases, and it disappears completely when money growth obeys the Friedman rule.

Figure 1 is also useful for a comparison of our model with previous search and money models. In this literature there are four types of meetings, represented by the corners of Figure 1, denoted  $A$ ,  $B$ ,  $C$ , and  $D$ . At  $B$  and  $D$  we have single-coincidence meetings, at  $C$  symmetric double-coincidence meetings, and at  $A$  no coincidence meetings. In this literature, traders attain the efficient outcome when they barter (point  $C$ ), and they exchange inefficient quantities when they trade money for goods (points  $B$  and  $D$ ). This suggests—that Figure 1 shows is not true—that money introduces inefficiencies into the bargaining that are not present in a barter economy.

<sup>8</sup> For example, consider a single-coincidence meeting with  $\varepsilon_i > 0$  and  $\varepsilon_j = 0$ . Then efficiency requires that agent  $j$  produce a positive quantity and receive nothing.



Our article is organized as follows. In Section 2 we present the environment. In Section 3 we derive and characterize the barter equilibrium, and in Section 4 the monetary equilibrium. Section 5 provides a discussion of the results, and Section 6 concludes.

## 2. THE ENVIRONMENT

We consider a random-matching model with divisible goods and divisible money along the lines of Shi (1999). The economy is populated with a large number of households, each consisting of a continuum of members of measure one who regard the household's utility as the common objective. In the market, household members attempt to exchange money or their production good for consumption goods. In this attempt, household members execute the strategies that have been given to them by their households. At the end of each period, the members pool their money holdings, which eliminates aggregate uncertainty for households. In the symmetric monetary equilibrium, the distribution of money holdings is degenerate across households. This facilitates the analysis, because we can focus on a representative household.<sup>10</sup> Finally, the utility of the household is defined as the sum of the utilities of its members.<sup>11</sup>

Although households differ in their preferences and production opportunities, they face the same decision problems, so that each household and each good can be treated symmetrically. In the following we refer to an arbitrary household as household  $h$ . Decision variables of this household are denoted by lowercase letters. Capital letters denote other households' variables, which are assumed to be given by the representative household  $h$ . Finally, variables corresponding to the next period are indexed by +1.

*2.1. Technology and Preferences.* The economy consists of a continuum of infinite-lived households and a continuum of goods, which are represented by points on the same circle of circumference 2, denoted by  $\mathcal{C}$ . The goods can also be viewed as different varieties of the same commodity. Each household is composed of a continuum of members who share the same technology and have the same preferences. Households are specialized in consumption and in production. Household  $h \in \mathcal{C}$  has the technology to produce good  $h$ . Producing  $q$  units of a good yields disutility  $c(q) = q$ . Goods cannot be stored, and production is instantaneous. Household members derive utility from consuming *all* goods other than their production good. The most preferred good of household  $h$  is  $h^*$ , chosen at

<sup>10</sup> The large-household assumption, extending a similar one in Lucas (1990), avoids difficulties that arise in models with a nondegenerate distribution of money holdings, and so allows for a tractable analysis of inflation. In search models of money, it was first used by Shi (1997, 1999, 2001). Lagos and Wright (2001) investigate an alternative assumption that yields a degenerate distribution of money holdings in random-matching models of divisible money.

<sup>11</sup> We have also introduced the large household in the indivisible-money, indivisible-goods model of Kiyotaki and Wright (1991, 1993) and the indivisible-money, divisible-goods model of Trejos and Wright (1995) and Shi (1995). Interestingly, we have found exactly the same reduced-form equations as in the original models (a proof of this claim is available on request).

random from  $\mathcal{C}$ . If we draw at random a commodity  $k$  from  $\mathcal{C}$ , the length  $l$  of the arc between  $k$  and  $h^*$  is uniformly distributed on  $[0, 1]$ . The function mapping the distance  $l$  and the quantity consumed  $q$  into utility is continuous in both arguments, strictly decreasing in  $l$ , and increasing in  $q$ . We adopt the following function:<sup>12</sup>

$$U(l, q) = \varepsilon(l)u(q)$$

The function  $\varepsilon(l)$  is strictly decreasing and twice differentiable, and it satisfies  $\varepsilon(0) = \varepsilon_{\text{sup}}$  and  $\varepsilon(1) = 0$ . The utility associated with the consumption of  $q$  units of the most preferred good is  $\varepsilon_{\text{sup}}u(q)$ , whereas the utility of consumption of the worst good is zero. Furthermore, we assume that  $u$  is increasing and twice differentiable, and satisfies  $u(0) = 0$ ,  $u'' < 0$ , and  $u'(0) = \infty$ . The probability that  $\varepsilon$  is less than  $x \in [0, \varepsilon_{\text{sup}}]$  for a good chosen at random is equal to

$$\mathbb{P}[\varepsilon(l) \leq x] = \mathbb{P}[l \geq \varepsilon^{-1}(x)] = 1 - \varepsilon^{-1}(x) \equiv F(x)$$

where  $F(\cdot)$  is a cumulative distribution with support  $[0, \varepsilon^{\text{sup}}]$  and density  $f$ .

Finally, the utility of a household in one period is the sum of the consumption utilities of its members minus their disutilities of production. The discount factor is  $\beta \in (0, 1)$ .

*2.2. Matching and Money.* Time is discrete. In each period each household member meets another member at random. Our assumptions about technology and preferences imply that in *each* match there is a double coincidence of real wants, i.e., each household is a consumer of the other household's production good.<sup>13</sup> In general, however, agents' preferences for each other's goods in a match are not symmetric. For an agent  $i$  matched with an agent  $j$ , the type of the match is given by the pair  $(\varepsilon_i, \varepsilon_j) \in [0, \varepsilon_{\text{sup}}] \times [0, \varepsilon_{\text{sup}}]$ . From the point of view of agent  $i$ 's partner, the match type is  $(\varepsilon_j, \varepsilon_i)$ . Generically,  $(\varepsilon_i, \varepsilon_j) \neq (\varepsilon_j, \varepsilon_i)$ . In the following, we adopt the following notation: For all  $\varepsilon = (\varepsilon_i, \varepsilon_j)$ ,  $\varepsilon' = (\varepsilon_j, \varepsilon_i)$ . Thus, if  $\varepsilon$  describes the match type from the point of view of agent  $i$ , then  $\varepsilon'$  describes the match type from the point of view of agent  $j$ . For any individual, the distribution of probabilities of the matches can be described by the measure  $\mu$  defined as follows:

$$\forall A \subseteq [0, \varepsilon_{\text{sup}}] \times [0, \varepsilon_{\text{sup}}], \quad \mu(A) = \int_0^{\varepsilon_{\text{sup}}} \int_0^{\varepsilon_{\text{sup}}} \mathbf{1}_A(x, y) f(x) f(y) dx dy$$

where  $\mathbf{1}_A(x, y)$  is an indicator function that is equal to 1 if  $(x, y) \in A$  and zero otherwise.

In the monetary economy, in addition to consumption goods, there is an intrinsically worthless, storable, and fully divisible object called fiat money. At the

<sup>12</sup> The analysis would also hold for any utility function satisfying  $\partial U(l, q)/\partial q > 0$ ,  $\partial U(l, q)/\partial l < 0$ ,  $\partial^2 U(l, q)/\partial q^2 < 0$ , and  $\partial^2 U(l, q)/\partial q \partial l < 0$ .

<sup>13</sup> In fact, there are single-coincidence meetings (along the segments  $[AB]$  and  $[AD]$  in Figure 1). The measure of single-coincidence meetings, however, is zero.

beginning of each period, each household has  $m$  units of money, which it divides evenly among its members, so that each member holds  $m$  units of money in a match. After this, household members are matched and carry out their exchanges according to the prescribed strategies. Within a period, no member can transfer money to another member of the same household. Thereafter, members bring back their receipts of money, and each agent consumes the goods he has bought.

*2.3. Terms of Trade.* The terms of trade are determined through bargaining games with alternating offers (Rubinstein, 1982). Two important features of the bargaining are that (i) bargaining strategies are determined at the household level but are carried out by household members, and (ii) bargaining strategies are determined ex ante for all possible match types, assuming that all future bargaining partners hold the same level of money holdings. This last feature is consistent with the equilibrium outcome where the distribution of money holdings is degenerate. Therefore, households' bargaining strategies are ex ante optimal. Finally, the equilibrium is fully characterized once we have described how households' members react if a partner unexpectedly shows up with a different level of money than what is expected in equilibrium. We assume that households' members play the strategies they have been instructed to play by their household prior to the matches, no matter what the level of money holdings of their partner in the match.<sup>14</sup>

In the following we consider the bargaining between agent  $i$  of household  $h$  and agent  $j$  from any other household. The bargaining proceeds as follows. Each period is divided into an infinite number of subperiods of length  $\Delta$ .<sup>15</sup> If, in a given subperiod, it is agent  $i$ 's turn to make an offer and agent  $j$  rejects the offer, then in the following subperiod it is agent  $j$ 's turn to make a counteroffer. If an offer is refused, the negotiation breaks down with probability  $\delta\Delta$  ( $\delta > 0$ ). If the negotiation breaks down, the players receive zero flow utility. Thus, our game is similar to the bargaining game where agents do not search while bargaining, as considered in Shi (1995) and Trejos and Wright (1995). The possibility of an exogenous breakdown of the negotiation gives an incentive to traders to agree immediately.<sup>16</sup>

In the alternating-offer game, offers and counteroffers converge to the same limiting proposal when  $\Delta$  goes to zero. Consequently, the first-mover advantage

<sup>14</sup> This pricing procedure differs from the procedure in Rauch (2000), who assumes that households' bargaining strategies are ex post optimal, because they are determined at the level of the household members. This alternative assumption would not affect our results qualitatively, but it would lead to more complicated expressions. See Berentsen and Rocheteau (2001) for a discussion of the implications of these different assumptions.

<sup>15</sup> Because the bargaining game takes place within a period, assuming a game with an infinite number of rounds is an approximation to a game with a large but finite number of rounds. This approximation is reasonable, because the Rubinstein game with a finite number of rounds converges to the Rubinstein game with an infinite horizon when the number of finite rounds increases without bounds.

<sup>16</sup> Instead of a breakdown of the bargaining, a cost of delaying an agreement could arise from the fact that households discount the future (Shi, 2001). Although formally equivalent, this last interpretation is not completely rigorous in a discrete-time model.

vanishes when  $\Delta$  goes to zero.<sup>17</sup> Because of this and because, as we will see, it facilitates the derivation of the dynamic equation describing the marginal utility of money, we let members of household  $h$  make the first offer in all meetings. In equilibrium all households have the same characteristics: as a consequence, first offers of household  $h$  are always accepted. Moreover, because the length of time between two consecutive offers is infinitesimal, the first offers are equal to the counteroffers that would have been made by  $h$ 's partners.

In the following, we will consider a barter economy and a monetary economy. In a barter economy, agents  $i$  and  $j$  bargain over the quantities  $q_\varepsilon^b$  and  $q_\varepsilon^s$  that are produced, consumed, and exchanged, where agent  $j$  ( $i$ ) produces commodity  $q_\varepsilon^b$  ( $q_\varepsilon^s$ ) and consumes commodity  $q_\varepsilon^s$  ( $q_\varepsilon^b$ ). The subscript  $\varepsilon$  indicates that the bargaining solution will depend on the type of match. In a monetary economy, they also bargain over a monetary transfer  $x_\varepsilon$ , where  $x_\varepsilon > 0$  means that agent  $i$  transfers  $x_\varepsilon$  units of money to agent  $j$ .

### 3. BARTER ECONOMY

In order to capture the benefit of money, we compare the allocation of resources of the barter economy with that of the monetary economy. In this section, we first present the barter equilibrium where money has no value.

3.1. *The Offers.* Consider the bargaining between agents  $i$  and  $j$ . Agent  $i$ , following the strategy prescribed by household  $h$ , proposes the terms of trade  $(q_\varepsilon^b, q_\varepsilon^s)$  if the match type is  $\varepsilon = (\varepsilon_i, \varepsilon_j)$ , i.e., if his valuation for the good produced by  $j$  is  $\varepsilon_i u(q_\varepsilon^b)$  and if  $j$ 's valuation for the good he produces is  $\varepsilon_j u(q_\varepsilon^s)$ . According to his offer,  $q_\varepsilon^b$  is the quantity he consumes and agent  $j$  produces, and  $q_\varepsilon^s$  is the quantity he produces and  $j$  consumes. If it is agent  $j$ 's turn to make an offer,  $j$  proposes the terms of trade  $(Q_{\varepsilon'}^b, Q_{\varepsilon'}^s)$ , where the index  $\varepsilon'$  indicates that, from the point of view of  $j$ , the match type is  $\varepsilon' = (\varepsilon_j, \varepsilon_i)$ . According to  $j$ 's offer,  $Q_{\varepsilon'}^s$  is the quantity  $i$  consumes and  $j$  produces, and  $Q_{\varepsilon'}^b$  is the quantity  $i$  produces and  $j$  consumes.

Denote by  $R_{\varepsilon'}$  the expected surplus (or reservation value) of agent  $j$  if he rejects the offer: it is taken as given by the household of agent  $i$ . Agent  $j$  will accept any offer that gives him a utility larger than  $R_{\varepsilon'}$ . Consequently, any optimal offer  $(q_\varepsilon^b, q_\varepsilon^s)$  must satisfy

$$(1) \quad \varepsilon_j u(q_\varepsilon^s) - q_\varepsilon^b = R_{\varepsilon'}$$

According to (1), agent  $i$ 's offer makes agent  $j$  just indifferent between accepting and rejecting the offer. The expected surplus of agent  $j$  equals

$$(2) \quad R_{\varepsilon'} = (1 - \delta\Delta)[\varepsilon_j u(Q_{\varepsilon'}^b) - Q_{\varepsilon'}^s]$$

<sup>17</sup>This argument is standard in the bargaining literature. See, for example, Muthoo (1999, Chapter 3), Osborne and Rubinstein (1990, Chapter 3).

If agent  $j$  rejects the offer, negotiations break down with probability  $\delta\Delta$ . With probability  $1 - \delta\Delta$  there is no breakdown and agent  $j$  makes the counteroffer  $(Q_{\varepsilon'}^b, Q_{\varepsilon'}^s)$  after a period of time of length  $\Delta$ .

3.2. *The Program of the Household.* A household's trading strategy consists of the terms of trade  $(q_\varepsilon^b, q_\varepsilon^s)$  for each  $\varepsilon \in E$ , and an acceptance rule that specifies whether an offer is acceptable or not. An offer is acceptable if it yields at least the reservation value  $R_\varepsilon$ , which is given by an equation that is similar to (2). To determine the offers for each period, the household chooses  $(q_\varepsilon^b, q_\varepsilon^s)_{\varepsilon \in E}$  to solve the following dynamic programming problem:

$$(3) \quad V = \max_{q_\varepsilon^b, q_\varepsilon^s} \left\{ \int_E [\varepsilon_i u(q_\varepsilon^b) - q_\varepsilon^s] d\mu(\varepsilon_i, \varepsilon_j) + \beta V \right\} \quad \text{s.t. (1)}$$

where  $V$  is the lifetime discounted utility of the household  $h$ . The right-hand side of Equation (3) has the following interpretation. For each match type  $\varepsilon$ , a member of household  $h$  makes the first proposal  $(q_\varepsilon^b, q_\varepsilon^s)$  that is immediately accepted by his partner. The net utility in an  $\varepsilon$ -meeting is then  $\varepsilon_i u(q_\varepsilon^b) - q_\varepsilon^s$ . Note that there is no state variable linking the present to the future. Consequently, for each period and each match type, households just maximize current utility of consumption net of current disutility of production. Replacing  $q_\varepsilon^b$  by its expression given by (1) and differentiating  $V$  with respect to  $q_\varepsilon^s$  yields the following first-order conditions:

$$(4) \quad \varepsilon_i u'(q_\varepsilon^b) = \frac{1}{\varepsilon_j u'(q_\varepsilon^s)} \quad \forall \varepsilon \in E$$

According to (4), the terms of trade  $(q_\varepsilon^b, q_\varepsilon^s)$  equalize the marginal utility of consumption,  $\varepsilon_i u'(q_\varepsilon^b)$ , to the marginal disutility of production times the quantity that must be produced to buy an additional unit of consumption good,  $1/\varepsilon_j u'(q_\varepsilon^s)$ . To see this, note that Equation (1) implies that  $\partial q_\varepsilon^s / \partial q_\varepsilon^b = 1/\varepsilon_j u'(q_\varepsilon^s)$ . Thus, to buy one additional unit of consumption, agent  $i$  must produce  $1/\varepsilon_j u'(q_\varepsilon^s)$  units of good for agent  $j$ , which costs  $1/\varepsilon_j u'(q_\varepsilon^s)$  in terms of utility.

3.3. *The Barter Equilibrium.* Given that all households have the same characteristics, it is natural to focus on a symmetric barter equilibrium, where all households apply the same trading strategies so that the values of the different variables of household  $h$  equal the values of the same variables of all other households. Consequently,  $(q_{\varepsilon'}^b, q_{\varepsilon'}^s) = (Q_{\varepsilon'}^b, Q_{\varepsilon'}^s)$  for all  $\varepsilon$ . Then, Equations (1) and (2) yield

$$(5) \quad \varepsilon_j u(q_\varepsilon^s) - q_\varepsilon^b = (1 - \delta\Delta) [\varepsilon_j u(q_{\varepsilon'}^b) - q_{\varepsilon'}^s]$$

DEFINITION 1. A symmetric barter equilibrium is a set of offers  $\{(q_\varepsilon^b, q_\varepsilon^s)\}_{\varepsilon \in E}$  satisfying Equations (4) and (5).

We want to compare the outcome of the decentralized economy to the allocation that a social planner would choose. The social planner treats all households symmetrically and consequently maximizes the utility of a representative household. In the Appendix we show that welfare is maximized if, for each match type  $\varepsilon \in E$ , the planner chooses the terms of trade  $(q_\varepsilon^b, q_\varepsilon^s)$  that exploit all the gains from trade, i.e., that maximize the total surplus of the match. For each match type, the total surplus equals  $\varepsilon_i u(q_\varepsilon^b) - q_\varepsilon^s + \varepsilon_j u(q_\varepsilon^s) - q_\varepsilon^b$ . Thus, total surplus is maximized if  $q_\varepsilon^b$  and  $q_\varepsilon^s$  satisfy

$$(6) \quad \varepsilon_i u'(q_\varepsilon^b) = \varepsilon_j u'(q_\varepsilon^s) = 1 \quad \forall \varepsilon \in E$$

Equation (6) simply states that in a match all the gains from trade are exploited if, for each good, marginal utility of consumption equals marginal disutility of production. Denote the efficient quantities by  $q_\varepsilon^{b*}$  and  $q_\varepsilon^{s*}$ , respectively.

In what follows we focus our attention on the limiting case when the length of time between two consecutive offers approaches zero, i.e.,  $\Delta \rightarrow 0$ . Then, the offers  $(q_\varepsilon^b, q_\varepsilon^s)$  and counteroffers  $(q_\varepsilon^s, q_\varepsilon^b)$  coincide, and the lifetime discounted utilities of all households are equal.

**PROPOSITION 1.** *Assume that  $\Delta \rightarrow 0$ . Then there is a unique symmetric barter equilibrium where in all asymmetric matches the quantities exchanged are inefficient and in all symmetric matches they are efficient. Furthermore, in an  $\varepsilon$ -meeting, the quantities traded  $(q_\varepsilon^b, q_\varepsilon^s)$  satisfy (4) and*

$$(7) \quad \frac{\varepsilon_j u(q_\varepsilon^s) - q_\varepsilon^b}{\varepsilon_i u(q_\varepsilon^b) - q_\varepsilon^s} = \frac{1}{\varepsilon_i u'(q_\varepsilon^b)}$$

Proposition 1 establishes the existence and uniqueness of the barter equilibrium.<sup>18</sup> In this equilibrium, traders exchange socially inefficient quantities when they have asymmetric tastes for each other’s goods, and efficient quantities when they have symmetric tastes. The reason for the observed inefficiencies is that in a barter economy the quantities produced and consumed determine simultaneously the size and the split of the total surplus of the match. Because households only care about their own surplus, the traders attain a bargaining solution that does not maximize the size of the surplus of the match.

The following corollary highlights the inefficiency of barter trades.

**COROLLARY 1.** *Consider an  $(\varepsilon_i, \varepsilon_j)$ -meeting in a barter economy with  $\varepsilon_i > \varepsilon_j$ . Then  $q_\varepsilon^b < q_\varepsilon^{b*}$ ,  $q_\varepsilon^s > q_\varepsilon^{s*}$ , and  $q_\varepsilon^b < q_\varepsilon^s$ . Furthermore, agent  $i$  receives more than half of the total surplus of the match.*

Corollary 1 establishes that if agent  $i$  likes agent  $j$ ’s good more, i.e., if  $\varepsilon_i > \varepsilon_j$ , then agent  $i$  gets less and agent  $j$  more than the efficient quantities, respectively.

<sup>18</sup> Note that the solution  $(q_\varepsilon^b, q_\varepsilon^s)$  of Equations (4) and (7) also maximizes the symmetric Nash product, i.e.,  $(q_\varepsilon^b, q_\varepsilon^s) = \arg \max [\varepsilon_i u(q_\varepsilon^b) - q_\varepsilon^s][\varepsilon_j u(q_\varepsilon^s) - q_\varepsilon^b]$ .

Moreover, agent  $i$ , who is more eager to consume than agent  $j$ , produces more and consumes less than his partner  $j$ , i.e.,  $q_\varepsilon^b < q_\varepsilon^s$ . This result illustrates in a drastic way the inefficiencies that arise in the barter economy. Indeed, although exploiting all the gains from trade requires that agent  $i$  consume more than he produces, i.e.,  $q_\varepsilon^{b*} > q_\varepsilon^{s*}$ , the noncooperative barter equilibrium attains just the opposite outcome. Finally, agent  $i$  receives more than half of the total surplus. One way to correct this inefficiency is to endow households with perfect knowledge of all past transactions. With this knowledge there are punishment strategies that implement the socially efficient outcome (Kocherlakota, 1998).

Finally, we want to emphasize that the barter economy displays a *double coincidence of real wants* in all matches. In each match, each household is a consumer of the other household's production, the traded quantities yield a positive surplus, and the exchange is quid pro quo. Nevertheless, in Section 4 we show that a monetary equilibrium exists and that money improves welfare. In fact, we show that the gains from trade are always higher in the monetary equilibrium.

#### 4. MONETARY ECONOMY

In this section, we consider the monetary economy and compare its equilibrium allocation of resources with that attained in the barter economy. The environment is exactly the same as in the barter economy, except that each member of each household holds  $m$  units of money when matched.

At the end of a period, the household receives a lump-sum money transfer  $\tau$ , which can be negative, and then carries the stock  $m_{t+1}$  to  $t + 1$ . Because the distribution of money holdings is degenerate in equilibrium, this will imply  $m_{t+1} = \gamma m$ , where  $\gamma$  is the gross growth rate of the money supply. We restrict  $\gamma$  to be larger than the discount factor  $\beta$ .<sup>19</sup> If  $V(m)$  denotes the lifetime discounted utility of household  $h$  endowed with  $m$  units of money, the household's marginal value of money is  $\omega = \beta V'(m_{t+1})$ , where  $V'(\cdot)$  is the derivative of  $V$  with respect to money holdings  $m$ . Denote by  $\Omega$  the marginal value of money of all other households.

**4.1. The Offers.** We again restrict our attention to meetings between member  $i$  of household  $h$  and some agent  $j$  from another household. If, in a given subperiod, it is agent  $i$ 's turn to make an offer, then agent  $i$ , following the strategy prescribed by his household, proposes the terms of trade  $(q_\varepsilon^b, q_\varepsilon^s, x_\varepsilon)$  if the match type is  $\varepsilon = (\varepsilon_i, \varepsilon_j)$ . As in the barter economy,  $q_\varepsilon^b$  is the quantity of goods produced by agent  $j$  and consumed by agent  $i$ , and  $q_\varepsilon^s$  is the quantity of goods delivered by agent  $i$ . The quantity  $x_\varepsilon$  is the amount of money exchanged. If  $x_\varepsilon > 0$ , agent  $i$  delivers  $x_\varepsilon$  units of money to agent  $j$ , and if  $x_\varepsilon < 0$ , he receives  $x_\varepsilon$  units of money. If it is agent

<sup>19</sup> When  $\gamma = \beta$ , there exists a continuum of monetary equilibria. In such equilibria, money serves purely as a store of value asymptotically. To focus on the transaction role of money, we restrict  $\gamma > \beta$ .

$j$ 's turn to make an offer, he proposes the terms of trade  $(Q_{\varepsilon'}^b, Q_{\varepsilon'}^s, X_{\varepsilon'})$ . From  $j$ 's point of view, the match type is  $\varepsilon' = (\varepsilon_j, \varepsilon_i)$ .

If agent  $j$  accepts the offer  $(q_{\varepsilon}^b, q_{\varepsilon}^s, x_{\varepsilon})$ , and if  $x_{\varepsilon} > 0$ , the acquired money balances  $x_{\varepsilon}$  will increase the stock of money that  $j$ 's household will hold at the beginning of the next period. Because each member is atomistic, the value of the amount of money obtained by member  $j$  is  $x_{\varepsilon}\Omega$ .<sup>20</sup> Therefore, the surplus of  $j$ 's household is equal to  $j$ 's consumption utility  $\varepsilon_j u(q_{\varepsilon}^s)$  minus  $j$ 's production disutility  $q_{\varepsilon}^b$  plus  $x_{\varepsilon}\Omega$ . Agent  $j$  accepts the offer if  $\varepsilon_j u(q_{\varepsilon}^s) - q_{\varepsilon}^b + x_{\varepsilon}\Omega \geq R_{\varepsilon'}$ , where  $R_{\varepsilon'}$  is the expected surplus of agent  $j$  if he rejects the offer. Thus, any optimal offer by agent  $i$  satisfies

$$(8) \quad \varepsilon_j u(q_{\varepsilon}^s) - q_{\varepsilon}^b + x_{\varepsilon}\Omega = R_{\varepsilon'}$$

The only difference between the barter offer (1) and the monetary offer (8) is that the monetary offer includes the monetary transfer  $x_{\varepsilon}$ . As in Section 3, one can derive an explicit expression for  $R_{\varepsilon'}$ . We find

$$(9) \quad R_{\varepsilon'} = (1 - \delta\Delta)[\varepsilon_j u(Q_{\varepsilon'}^b) - Q_{\varepsilon'}^s - X_{\varepsilon'}\Omega]$$

where  $(Q_{\varepsilon'}^b, Q_{\varepsilon'}^s, X_{\varepsilon'})$  is agent  $j$ 's proposal when it is his turn to make an offer. Note that  $X_{\varepsilon'} > 0$  is a monetary transfer from  $j$  to  $i$ .

4.2. *The Program of the Household.* When the household determines the terms of trade, it is subject to two sets of constraints. First, household members cannot spend more money than what they have:

$$(10) \quad x_{\varepsilon} \leq m \quad \forall \varepsilon \in E$$

Second, household members cannot ask for more money than what their bargaining partner holds:

$$(11) \quad -x_{\varepsilon} \leq M \quad \forall \varepsilon \in E$$

A household's trading strategy consists of the terms of trade  $(q_{\varepsilon}^b, q_{\varepsilon}^s, x_{\varepsilon})$  for each  $\varepsilon \in E$ , and an acceptance rule for each offer  $(Q_{\varepsilon'}^b, Q_{\varepsilon'}^s, X_{\varepsilon'})$  by another household. Again, agent  $i$  from household  $h$  makes the first offer, which is immediately accepted by  $j$ . For each period, the household chooses  $\{m_{+1}, (q_{\varepsilon}^b, q_{\varepsilon}^s, x_{\varepsilon})_{\varepsilon \in E}\}$  to solve

<sup>20</sup> To see why, suppose that the measure of a member is  $\mu$ . Then for the household, the value of  $x$  additional units of money received by a member is  $\beta[V(m_{+1} + x\mu) - V(m_{+1})]$ . To express the value of  $x$  additional units of money for a member, we must multiply this quantity by the scale factor  $1/\mu$ . Because members are atomistic, we let  $\mu \rightarrow 0$  to get  $\lim_{\mu \rightarrow 0} \beta[V(m_{+1} + x\mu) - V(m_{+1})]/\mu = x\beta V'(m_{+1}) = x\omega$ . Thus, from the point of view of the household,  $x\omega$  is a member's indirect utility of receiving  $x$  units of money in a match.

the following dynamic programming problem:

$$(12) \quad V(m) = \max_{q_\varepsilon^b, q_\varepsilon^s, x_\varepsilon, m_{+1}} \left\{ \int_E [\varepsilon_i u(q_\varepsilon^b) - q_\varepsilon^s] d\mu(\varepsilon_i, \varepsilon_j) + \beta V(m_{+1}) \right\}$$

subject to the constraints (8), (10), (11), and

$$(13) \quad m_{+1} - m = \tau - \int_E x_\varepsilon d\mu(\varepsilon_i, \varepsilon_j)$$

The variables taken as given in the above problem are the state variable  $m$  and other households' choices (the uppercase variables). The integral in Equation (12) aggregates the net utilities in all meetings. Equation (13) specifies the law of motion of the household's money balances. The first term on the right-hand side specifies the additional currency the household receives each period. The second term is the net amount of money received by the household.

Denote by  $\lambda_\varepsilon$  the multipliers associated with constraints (10). The multipliers associated with constraints (11) will be denoted by  $\pi_\varepsilon$ . Then, the program of the household can be rewritten as follows:

$$V(m) = \max_{q_\varepsilon^b, q_\varepsilon^s, m_{+1}} \left\{ \int_E [\varepsilon_i u(q_\varepsilon^b) - q_\varepsilon^s] d\mu(\varepsilon_i, \varepsilon_j) + \int_E \lambda_\varepsilon (m - x_\varepsilon) d\mu(\varepsilon_i, \varepsilon_j) \right. \\ \left. + \int_E \pi_\varepsilon (M + x_\varepsilon) d\mu(\varepsilon_i, \varepsilon_j) + \beta V(m_{+1}) \right\}$$

where  $m_{+1}$  is given by Equation (13). Furthermore, according to (8),  $x_\varepsilon$  can be expressed as a function of  $q_\varepsilon^s$  and  $q_\varepsilon^b$ . The first-order conditions and the envelope condition are as follows:

$$(14) \quad \varepsilon_i u'(q_\varepsilon^b) = \frac{\lambda_\varepsilon - \pi_\varepsilon + \omega}{\Omega} \quad \forall \varepsilon \in E$$

$$(15) \quad \varepsilon_j u'(q_\varepsilon^s) = \frac{\Omega}{\omega + \lambda_\varepsilon - \pi_\varepsilon} \quad \forall \varepsilon \in E$$

$$(16) \quad \lambda_\varepsilon (m - x_\varepsilon) = 0 \quad \forall \varepsilon \in E$$

$$(17) \quad \pi_\varepsilon (M + x_\varepsilon) = 0 \quad \forall \varepsilon \in E$$

$$(18) \quad \frac{\omega_{-1}}{\beta} = \int_E [\omega + \lambda_\varepsilon] d\mu(\varepsilon_i, \varepsilon_j)$$

Let us first interpret the optimal choices of  $q_\varepsilon^b$  and  $q_\varepsilon^s$ . For each match type, the household has two decisions. First, how much to consume. Second, how to finance this current consumption—through sale of money, production, or both. The first

decision implies the equalization of the marginal utility of consumption to the marginal cost of financing this consumption. The second decision is an *arbitrage* condition. This condition requires that the value of the amount of money that the household must pay be equal to the marginal cost of producing additional goods to finance this purchase.

To see the relationship between the two choices, note that the first-order conditions (14) and (15) imply the following relation:

$$(19) \quad \varepsilon_i u'(q_\varepsilon^b) = \frac{\lambda_\varepsilon - \pi_\varepsilon + \omega}{\Omega} = \frac{1}{\varepsilon_j u'(q_\varepsilon^s)} \quad \forall \varepsilon \in E$$

The left-hand side of (19) is the marginal utility of consumption. The middle expression is the monetary cost of financing one additional unit of the consumption good with money. The right-hand side is the marginal cost of financing this unit through production.<sup>21</sup> The equality on the right-hand side of (19) is the arbitrage condition: It requires that these two costs be equal. To interpret the monetary cost, note from Equation (8) that to buy an additional unit of the consumption good, the household must give up  $1/\Omega$  units of money. Since money has a marginal discounted value  $\omega$  in the future, the exchange has an opportunity cost  $\omega/\Omega$ . In addition, the exchange may make the trading constraint (10) more binding and (11) less binding, thus generating a shadow cost  $(\lambda_\varepsilon - \pi_\varepsilon)/\Omega$ . The sum of the opportunity cost of money and the shadow cost is the monetary cost of financing one additional unit of the consumption good.<sup>22</sup>

Equations (16) and (17) are the Kuhn–Tucker conditions associated with the multipliers  $\lambda_\varepsilon$  and  $\pi_\varepsilon$ , respectively. Note from Equations (14) and (15) that if the constraints on money holdings do not bind, i.e., if  $\lambda_\varepsilon = \pi_\varepsilon = 0$ , the quantities produced and exchanged are socially efficient.

Finally, Equation (18) describes the evolution of the marginal value of money. It states that the marginal value of money today,  $\omega_{-1}/\beta$ , equals the marginal benefit of money in all meetings,  $\int_E [\omega + \lambda_\varepsilon] d\mu(\varepsilon_i, \varepsilon_j)$ . This integral has the following interpretation: In an  $\varepsilon$ -meeting the value of an additional unit of money is equal to  $\omega + \lambda_\varepsilon$ , which is simply  $\omega$  if the agent's cash constraint is not binding. Note that the value of money is larger in a meeting where the agent's cash constraint is binding, because an additional unit of money would allow the agent to reduce inefficient production.

**4.3. The Symmetric Equilibrium.** Given that all households have the same characteristics, it is natural to focus on a symmetric equilibrium, where all households apply the same trading strategies so that the values of the different variables of the household  $h$  equal the values of the same variables of all other households. In

<sup>21</sup> Equation (8) implies that an additional unit of the consumption good requires  $1/[\varepsilon_j u'(q_\varepsilon^s)]$  units of production to exchange for.

<sup>22</sup> Note that the household is willing to produce at a higher marginal cost to finance additional consumption when the constraint (10) is binding. In contrast, if (11) is binding, the household has to pay less. The shadow price  $\pi_\varepsilon$ , therefore, can be interpreted as a discount that the other household is willing to sacrifice to obtain additional money if the constraint (11) is binding.

a symmetric equilibrium, uppercase variables and lowercase variables are equal:  $\omega = \Omega$ ,  $m = M$ , and  $(q_{\varepsilon'}^b, q_{\varepsilon'}^s, x_{\varepsilon'}) = (Q_{\varepsilon'}^b, Q_{\varepsilon'}^s, X_{\varepsilon'})$  for all  $\varepsilon$ . Then, Equations (8) and (9) yield

$$(20) \quad \varepsilon_j u(q_\varepsilon^s) - q_\varepsilon^b + x_\varepsilon \omega = (1 - \delta \Delta) [\varepsilon_j u(q_{\varepsilon'}^b) - q_{\varepsilon'}^s - x_{\varepsilon'} \omega]$$

DEFINITION 2. A symmetric monetary steady-state equilibrium is an  $\omega > 0$  and a set  $\{(q_\varepsilon^b, q_\varepsilon^s, x_\varepsilon, \lambda_\varepsilon, \pi_\varepsilon)\}_{\varepsilon \in E}$  such that Equations (14)–(18) and (20) hold.

We again want to compare the outcome of the decentralized economy with the allocation that a social planner would choose in order to maximize social welfare. As in the barter economy, welfare is maximized if in each meeting  $q_\varepsilon^b$  and  $q_\varepsilon^s$  satisfy  $\varepsilon_i u'(q_\varepsilon^b) = \varepsilon_j u'(q_\varepsilon^s) = 1$ . Lemma 1 characterizes the terms of trade in the monetary equilibrium when the length of time between two consecutive offers approaches zero.

LEMMA 1. Assume that  $\Delta \rightarrow 0$ , and consider an  $(\varepsilon_i, \varepsilon_j)$ -meeting. Then there are two cases:

Case a. One of the traders' constraints on money holdings is binding. If agent  $i$ 's constraint on money holding is binding, then  $q_\varepsilon^b < q_\varepsilon^{b*}$ ,  $q_\varepsilon^s > q_\varepsilon^{s*}$ ,  $x_\varepsilon = m$ , and agent  $i$  receives more than one-half of the total surplus of the match. The exchanged quantities  $q_\varepsilon^b$  and  $q_\varepsilon^s$  satisfy equations (19) and

$$(21) \quad \frac{1}{\varepsilon_i u'(q_\varepsilon^b)} = \frac{\varepsilon_j u(q_\varepsilon^s) - q_\varepsilon^b + m\omega}{\varepsilon_i u(q_\varepsilon^b) - q_\varepsilon^s - m\omega}$$

Case b. No trader is constrained by his money holdings. Then  $q_\varepsilon^b = q_\varepsilon^{b*}$ ,  $q_\varepsilon^s = q_\varepsilon^{s*}$ , and  $x_\varepsilon$  satisfies the following splitting rule:

$$(22) \quad \frac{\varepsilon_j u(q_\varepsilon^{s*}) - q_\varepsilon^{b*} + x_\varepsilon \omega}{\varepsilon_i u(q_\varepsilon^{b*}) - q_\varepsilon^{s*} - x_\varepsilon \omega} = 1$$

According to case b of Lemma 1, if no agent is constrained by his money holdings, then the matched agents produce, exchange, and consume efficient quantities, and they split the total surplus of the match evenly. If the constraint on money holdings of one of the agents is binding (case a of Lemma 1), then the terms of trade are inefficient and the agent whose constraint is binding receives more than half of the total surplus. Note that in Lemma 1 we do not mention the case when agent  $j$ 's constraint on money holding is binding, because by symmetry we would have  $q_\varepsilon^b > q_\varepsilon^{b*}$ ,  $q_\varepsilon^s < q_\varepsilon^{s*}$ ,  $x_\varepsilon = -m$ .

To see why monetary trades, in contrast to barter trades, can generate efficient allocations even in asymmetric matches, consider a monetary match between agents  $i$  and  $j$ , with  $\varepsilon_i > \varepsilon_j$ . Because for a given quantity  $q$   $i$ 's marginal utility of consumption is higher than  $j$ 's, efficiency requires that agent  $j$  produces a larger quantity than agent  $i$ . Agent  $j$  will agree to such an arrangement if agent  $i$  is able to

compensate him by transferring sufficient claims to future consumption (money). If  $i$ 's constraint on money holding is not binding, he is able to do so, and  $i$  and  $j$  exchange efficient quantities. If not, the bargaining will result in inefficient quantities produced and exchanged.

Money, in contrast to real production, is a perfect device to transfer utility, because for all households an additional unit of money has the same real value  $\omega$ .<sup>23</sup> Real production, on the other hand, is an imperfect means to transfer utility, because marginal utility for the consumer and marginal cost of the producer vary with the quantity produced and exchanged. We want to emphasize the importance, for efficiency, of money being equally valued. If money were not equally valued, according to (14), inefficient quantities would be produced and exchanged even if constraints on money holdings were not binding ( $\lambda_\varepsilon = \pi_\varepsilon = 0$ ). For example, if  $\omega > \Omega$ , which would mean that household  $h$  valued money more than other households, household  $h$  would receive less and produce more than the efficient quantities.

Finally, Lemma 1 implies that  $x_\varepsilon = -x_{\varepsilon'}$ . Accordingly, because the distribution of match types is symmetric (i.e., the probability density of the match  $(\varepsilon_i, \varepsilon_j)$  is equal to the probability density of the match  $(\varepsilon_j, \varepsilon_i)$ ), the net amount of money paid by a household to other households is zero, i.e.,

$$(23) \quad \int_E x_\varepsilon d\mu(\varepsilon_i, \varepsilon_j) = 0$$

Equation (23) corresponds to the money-market equilibrium condition.

4.4. *Reduced-Form Equations.* Lemma 1 and the envelope condition (18) allow us to reduce the model to two equations that can be easily interpreted. Moreover, the reduced-form equations can be related to the search models of indivisible money of Shi (1995) and Trejos and Wright (1995). For a given marginal value of money  $\omega$ , the terms of trade satisfy

$$(24) \quad (q_\varepsilon^b, q_\varepsilon^s, x_\varepsilon) = \arg \max [\varepsilon_i u(q_\varepsilon^b) - q_\varepsilon^s - x_\varepsilon \omega] [\varepsilon_j u(q_\varepsilon^s) - q_\varepsilon^b + x_\varepsilon \omega] \\ \text{s.t. } -m \leq x_\varepsilon \leq m$$

For given terms of trade  $(q_\varepsilon^b, q_\varepsilon^s, x_\varepsilon)$ , the marginal value of money satisfies

$$(25) \quad \omega_{-1} = \beta \int_E \max [\varepsilon_i u'(q_\varepsilon^b) \omega, \omega] d\mu(\varepsilon_i, \varepsilon_j)$$

Thus, as in Shi (1995) and Trejos and Wright (1995), the terms of trade satisfy the axiomatic Nash solution (see Equation (24)). In contrast to their models, however,

<sup>23</sup> All households have the same marginal value of money because the distribution of money is degenerate. In models with a nondegenerate distribution of money (e.g., Berentsen, 2000; Rocheteau, 2000; and Zhou, 1999), the marginal value of money differs among agents.

the bargaining partners decide not only how much to produce and consume, but also how much money is transferred in the match.

The envelope condition (25) is very similar to the Bellman equation that characterizes the value of being a buyer in Shi (1995) and Trejos and Wright (1995). It has the following interpretation. For the household, the value of an additional unit of money received at the end of the previous period,  $\omega_{-1}$ , depends on the use of this unit of money in the current period. In the current period, this unit of money can be either spent or saved. If it is spent in an  $\varepsilon$ -meeting, the value of this unit is  $\varepsilon_i u'(q_\varepsilon^b)\omega$ , because an additional unit of money buys  $\omega$  units of good, which when consumed generate  $\varepsilon_i u'(q_\varepsilon^b)$  additional utility. If the unit of money is saved, its value is simply  $\omega$ . Note that an additional unit of money is spent if and only if the marginal utility of consumption is larger than the marginal value of money, i.e., if  $\varepsilon_i u'(q_\varepsilon^b)\omega > \omega$ .

4.5. *Inefficiencies and the Real Value of Money.* In the following we want to characterize the set of match types that generate inefficient trades. To derive this set, we introduce the distance function  $\mathcal{D}(\varepsilon_i, \varepsilon_j)$ , which measures the asymmetry in preferences in a match. It is defined as follows:<sup>24</sup>

$$(26) \quad \mathcal{D}(\varepsilon_i, \varepsilon_j) = \left| \frac{\varphi(\varepsilon_i) - \varphi(\varepsilon_j)}{2} \right|$$

where  $\varphi(\varepsilon) = \varepsilon u(u'^{-1}(\frac{1}{\varepsilon})) + u'^{-1}(\frac{1}{\varepsilon})$ . The term  $\varepsilon u(u'^{-1}(\frac{1}{\varepsilon}))$  is the consumption utility of an agent with taste  $\varepsilon$  who consumes the efficient quantity  $q_\varepsilon^{b*} = u'^{-1}(\frac{1}{\varepsilon})$ , defined in (6), and the term  $u'^{-1}(\frac{1}{\varepsilon})$  is the cost of producing this quantity. Note that  $\varphi(\varepsilon_i) - \varphi(\varepsilon_j)$  is the surplus of agent  $i$  minus the surplus of agent  $j$  when both agents consume and produce efficient quantities.

PROPOSITION 2. *There is a  $\bar{\gamma} > \beta$  such that if  $\gamma \notin (\beta, \bar{\gamma})$ , no monetary steady-state equilibrium exists, and if  $\gamma \in (\beta, \bar{\gamma})$ , a unique symmetric monetary steady-state equilibrium with the following properties exists:*

- (i) *An  $(\varepsilon_i, \varepsilon_j)$ -trade is inefficient if and only if  $\mathcal{D}(\varepsilon_i, \varepsilon_j) > m\omega$ .*
- (ii) *The measure of inefficient trades is decreasing in  $m\omega$  and tends to zero as  $\gamma \rightarrow \beta$ . The upper limit for the gross growth rate of the money supply satisfies*

$$(27) \quad \bar{\gamma} = \beta \int_E \max(\varepsilon_i u'(\hat{q}_\varepsilon^b), 1) d\mu(\varepsilon_i, \varepsilon_j)$$

where  $\hat{q}_\varepsilon^b$  is the quantity consumed by an individual in an  $\varepsilon$ -meeting in the barter economy.

<sup>24</sup> Note that  $\mathcal{D}(\varepsilon_i, \varepsilon_j)$  satisfies the three properties of a distance: (i)  $\mathcal{D}(\varepsilon_i, \varepsilon_j) \geq 0$  for all  $(\varepsilon_i, \varepsilon_j)$ . Furthermore,  $\mathcal{D}(\varepsilon_i, \varepsilon_j) = 0$  only when  $\varepsilon_i = \varepsilon_j$ . (ii)  $\mathcal{D}(\varepsilon_i, \varepsilon_j) = \mathcal{D}(\varepsilon_j, \varepsilon_i)$ . (iii)  $\mathcal{D}(\varepsilon_i, \varepsilon_j) \leq \mathcal{D}(\varepsilon_i, \varepsilon_k) + \mathcal{D}(\varepsilon_k, \varepsilon_j)$ .

Proposition 2 establishes the existence of a unique symmetric monetary steady-state equilibrium if the gross growth rate of the money supply  $\gamma$  is larger than the discount factor but not too high. If  $\gamma > \bar{\gamma}$ , fiat money is valueless and agents produce and consume the barter quantities.

In contrast to barter trades, monetary trades can generate efficient allocations in asymmetric matches. Whether a monetary trade is inefficient depends on the degree of asymmetry of the match, as measured by the distance  $\mathcal{D}$ , and the real stock of money  $m\omega$  (see Figure 1).<sup>25</sup> This result implies three things. First, it is the degree of asymmetry for each other's goods that matters for efficiency. Second, if the real stock of money is large, a randomly chosen match type is less likely to generate inefficient terms of trade. In other words, the measure of inefficient monetary trades decreases with increasing real value of money. Third, since the real stock of money depends negatively on the gross growth rate of the money supply ( $\gamma$ ), money growth reduces efficiency, and hence money is not superneutral. In contrast, the Friedman rule is socially efficient because it maximizes the real stock of money.

Finally, note that when the gross growth rate of the money supply approaches the upper bound  $\bar{\gamma}$ , the allocation of resources in the monetary economy converges to the (mis)allocation of the corresponding barter economy. According to (27), the value of the upper limit  $\bar{\gamma}$  depends on the size of inefficiencies in the barter economy. If in the barter economy terms of trade are very inefficient, in the sense that in a large fraction of trades  $\varepsilon_i u'(\hat{q}_\varepsilon^b)$  is large relative to the marginal cost of production, then a monetary equilibrium exists even for high inflation rates. For example, in an environment with a positive measure of single-coincidence meetings,  $\bar{\gamma} = +\infty$ . To see this, note that in single-coincidence meeting with  $\varepsilon_i > 0$  and  $\varepsilon_j = 0$ ,  $\varepsilon_i u'(\hat{q}_\varepsilon^b) = +\infty$ . Consequently, with a positive measure of single-coincidence meetings, a monetary equilibrium exists for any finite growth rate of the money supply. In contrast, in a double-coincidence environment with symmetric preferences, in all meetings  $\varepsilon_i = \varepsilon_j$  and  $\varepsilon_i u'(\hat{q}_\varepsilon^b) = 1$ , which implies that  $\bar{\gamma} = \beta$ . Consequently, in symmetric double-coincidence-of-real-wants environments money cannot be valued.

## 5. DISCUSSION

In the following we want to develop more intuition for the results presented in the previous sections. To do so, we use the fact that the solution of Rubinstein's bargaining game between the households of agents  $i$  and  $j$  maximizes the symmetric Nash product of the traders' surpluses  $S_i$  and  $S_j$ , respectively. In what follows we assume, without loss of generality, that  $\varepsilon_i > \varepsilon_j$ . Bargaining solutions and figures are drawn for a single  $\varepsilon \in E$  only. Note that the figures are not just a heuristic exposition but a precise depiction of the households' behavior.

Consider, first, the barter economy where the surpluses of agents  $i$  and  $j$  are  $S_i = \varepsilon_i u(q_\varepsilon^b) - q_\varepsilon^s$  and  $S_j = \varepsilon_j u(q_\varepsilon^s) - q_\varepsilon^b$ , respectively. The Pareto frontier of the

<sup>25</sup> According to (8), the amount of money that must be spent to buy one additional unit of a good is  $\frac{1}{\omega}$ . Hence,  $m\omega$  can be viewed as a proxy for the real stock of money.

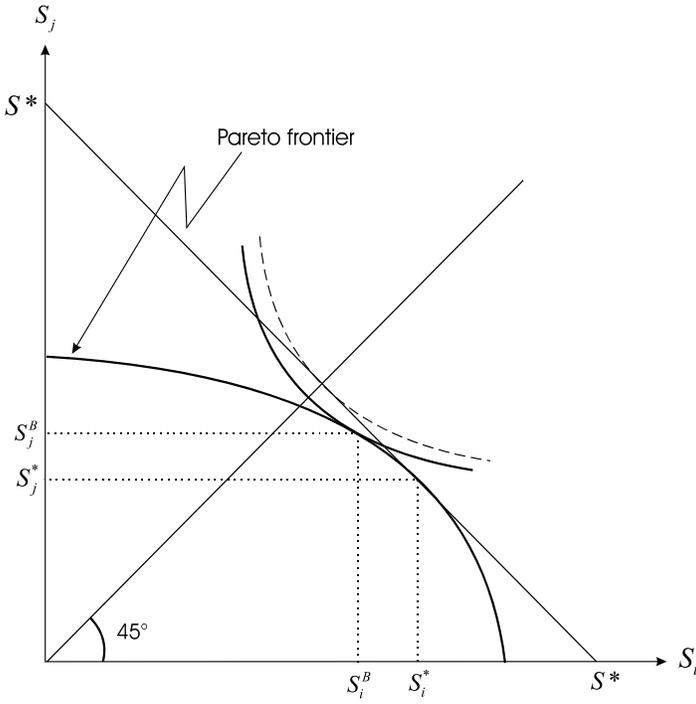


FIGURE 2

SOCIALLY INEFFICIENT BARTERING

bargaining set is the concave curve in Figure 2.<sup>26</sup> Each point on this curve uniquely determines what quantities of goods  $i$  and  $j$  produce, exchange, and consume. The bargaining solution is determined by the tangency point between the Pareto frontier and a Nash product curve (the convex curves in Figure 2). The surpluses of the players at this point are denoted by  $(S_i^B, S_j^B)$ . Note that  $(S_i^B, S_j^B)$  is located to the right of the 45-degree line, which means that  $i$  gets more than half of the surplus (see Corollary 1).

The line  $S^*S^*$  represents every possible split  $(S_i, S_j)$  of the maximal total surplus of the match  $S^* = S_i^* + S_j^*$  where  $S_i^* = \varepsilon_i u(q_e^{b*}) - q_e^{s*}$  and  $S_j^* = \varepsilon_j u(q_e^{s*}) - q_e^{b*}$ . Consequently,  $S^*S^*$  is the Pareto frontier of the same bargaining game with transferable utility. A bargaining game is said to have transferable utility if there is a device that allows a player to decrease his own payoff and increase that of a partner by the same amount. A two-person bargaining game with transferable utility is fully characterized by the disagreement payoffs of the two players and by the total transferable wealth ( $S^*$ ) available to the players  $i$  and  $j$ .

Bartering is socially inefficient because  $(S_i^B, S_j^B) \notin S^*S^*$ , which means that the terms of trade do not exploit all the gains from trade, i.e., they do not maximize

<sup>26</sup> See the Appendix for a formal derivation of the Pareto frontier and some of the other technical details.

the total surplus of the match. Note that  $\varepsilon_i > \varepsilon_j$  implies  $S_i^* > S_j^*$  and that if the asymmetry in preferences is large,  $S_j^*$  may be negative. Moreover, at  $(S_i^*, S_j^*)$  the slope of the Pareto frontier is  $-1$ . Hence,  $(S_i^*, S_j^*)$  is at the tangency point between the Pareto frontier and the line  $S^*S^*$ .

In the barter economy agents have no device to transfer utility, which implies that  $(S_i^*, S_j^*)$  is the only feasible split of the maximal total surplus  $S^*$ . If, however, the traders had a device to transfer utility, they could attain any point on  $S^*S^*$  by producing and consuming efficient quantities, and then split the total utility  $S^*$  through a transfer of utility. Note that the Nash product curve (see the dotted curve in Figure 2) has a tangency at the line  $S^*S^*$  where this line crosses the 45-degree line. Consequently, if agents could transfer utility, they would produce and exchange efficient quantities and then exchange utility to attain  $S_i = S_j = S^*/2$ .

We next discuss how the Pareto frontier is transformed when money is introduced into the economy. To do so, we consider a household's trade decisions for one particular match, where a member of the household (agent  $i$ ) meets a member of another household (agent  $j$ ).<sup>27</sup> In the monetary economy, the surpluses of agents  $i$  and  $j$  are  $S_i^M = \varepsilon_i u(q_\varepsilon^b) - q_\varepsilon^s - x_\varepsilon \omega$  and  $S_j^M = \varepsilon_j u(q_\varepsilon^s) - q_\varepsilon^b + x_\varepsilon \omega$ , respectively. The transformation is displayed in Figure 3, where the Pareto frontier of the barter economy is the dotted concave curve and the Pareto frontier of the monetary economy is the solid concave curve. There is again the line  $S^*S^*$  that represents every possible split  $(S_i, S_j)$  of the maximal total surplus of the match.

To see how the transformation works, consider point  $A$  in Figure 3, which lies on the Pareto frontier of the barter economy and the associated segment  $[A', A'']$ . Point  $A$  uniquely determines what quantities of goods agents  $i$  and  $j$  produce and exchange. The segment  $[A', A'']$  represents the set of surpluses that  $i$  and  $j$  can reach by transferring money without changing these quantities. At  $A'$  [ $A''$ ] player  $j$  [ $i$ ] receives  $m$  units of money, so that  $A' = (S_i - m\omega, S_j + m\omega)$  [ $A'' = (S_i + m\omega, S_j - m\omega)$ ]. Note that the slope of the segment  $[A', A'']$  is  $-1$ , because households have the same constant marginal value of money.

As drawn in Figure 3, only point  $A'$  lies on the Pareto frontier of the bargaining set in the monetary economy. This is true for any point on the Pareto frontier of the barter economy that satisfies  $S_i < S_i^*$ . In contrast, if  $S_i > S_i^*$ , the point  $A'' = (S_i + m\omega, S_j - m\omega)$  would lie on the Pareto frontier of the monetary economy. If  $S_i = S_i^*$ , the points  $A' = (S_i - m\omega, S_j + m\omega)$  and  $A'' = (S_i + m\omega, S_j - m\omega)$ , and consequently the whole segment  $[A' A'']$ , would lie on the Pareto frontier of the bargaining set of the monetary economy. Note that in this case, the segment  $[A' A'']$  lies on the line  $S^*S^*$ .

If agents had a device to transfer utility, they could attain all points on the line  $S^*S^*$ . In the monetary economy, divisible money is such a device, in the sense that agents can attain all points on  $S^*S^*$  satisfying  $S_i \in [S_i^* - m\omega; S_i^* + m\omega]$ . However, money does not transfer utility perfectly, because the Pareto frontier in the

<sup>27</sup> Since this particular match is one of many trades that the household experiences in the period, the household's decisions for this match have negligible influence on the household's total stock of money and the marginal value of money,  $\omega$ . Thus, in this exposition,  $m\omega$  is taken as given.

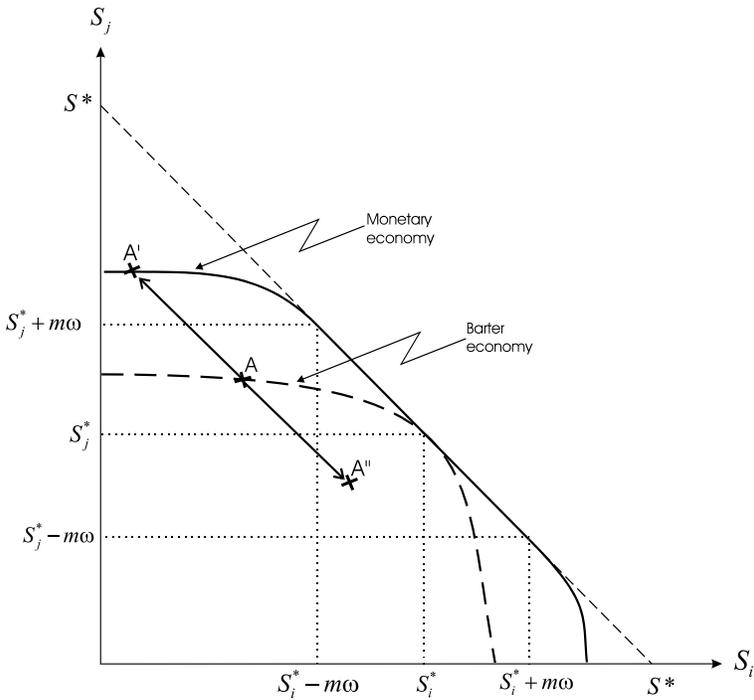


FIGURE 3

PARETO FRONTIER WITH AND WITHOUT MONEY

monetary economy does not fully coincide with  $S^*S^*$ . The extent to which money transfers utility perfectly depends positively on the real value of money holdings  $m\omega$ : A higher real value of money corresponds to a larger part of the Pareto frontier that fully coincides with  $S^*S^*$ .

Note that under the Friedman rule for all types of matches, we have  $S^*/2 \in [S_i^* - m\omega; S_i^* + m\omega]$ . Thus, under the Friedman rule the real value of money is high enough to allow agents to split evenly the total surplus of all matches when socially efficient quantities are produced. Consequently, in all matches all gains from trade are realized.

An important point to note from Figure 3 is that the set of agreements that can be reached in the monetary economy contains the set of agreements that can be reached in the barter economy.<sup>28</sup> Thus, with valued money the total surplus in a match can never be lower than the total surplus attained in the corresponding barter economy. Consequently, welfare in a monetary economy

<sup>28</sup>Note that money does not enlarge the set of feasible transactions; traders can exchange any quantity of goods, with or without money. The benefit of money arises because it allows production and exchange of goods that would not satisfy the quid pro quo requirement without a transfer of money. For a similar argument consider Ostroy (1973).

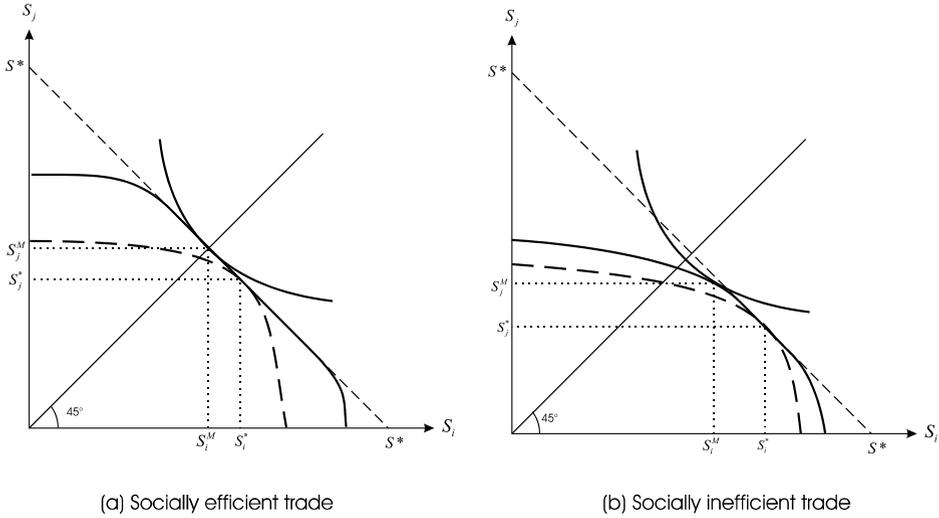


FIGURE 4

EFFICIENT AND INEFFICIENT TRADES IN THE MONETARY ECONOMY

must be *at least* as large as in the corresponding barter economy. This result is similar to results obtained in the theory of international trade, where allowing trade is never welfare-decreasing relative to autarky, simply because countries have always the option not to trade. In our model, agents can always choose to barter, which implies that introducing valued fiat money can never decrease welfare.

This last point is even more general: If we compare two monetary economies with respective gross growth rates of the money supply  $\gamma_1 > \gamma_2$ , we have  $(m\omega)_2 > (m\omega)_1$ . Consequently, the set of agreements that can be reached in the monetary economy with rate  $\gamma_2$  contains the set of agreements that can be reached in the monetary economy with rate  $\gamma_1$ . Thus, welfare in a monetary economy with rate  $\gamma_2$  must be *at least* as large as in an monetary economy with rate  $\gamma_1$ . This explains intuitively how the welfare-improving role of money is linked to the real value of money holdings.

In the monetary economy traded quantities are efficient if the asymmetry in preferences measured by the distance  $\mathcal{D}(\cdot, \cdot)$  defined in Equation (26) is lower than the real value of money. Figure 4 displays an asymmetric match where efficient quantities are exchanged (a) and an asymmetric match where inefficient quantities are traded (b). The dotted curves are the Pareto frontiers of the bargaining set in the barter economy. The real value of money  $(m\omega)$  is larger in Figure 4(a) than in Figure 4(b), because the segment of the Pareto frontier that lies on the line  $S^*S^*$  is larger on the left. Remember that the real value of money depends on the rate of growth of the money supply.

In Figure 5 we show how the distance  $\mathcal{D}(\varepsilon_i, \varepsilon_j)$  can be represented in the graphs of the previous figures. A trade is socially inefficient if the surplus of  $j$  when he

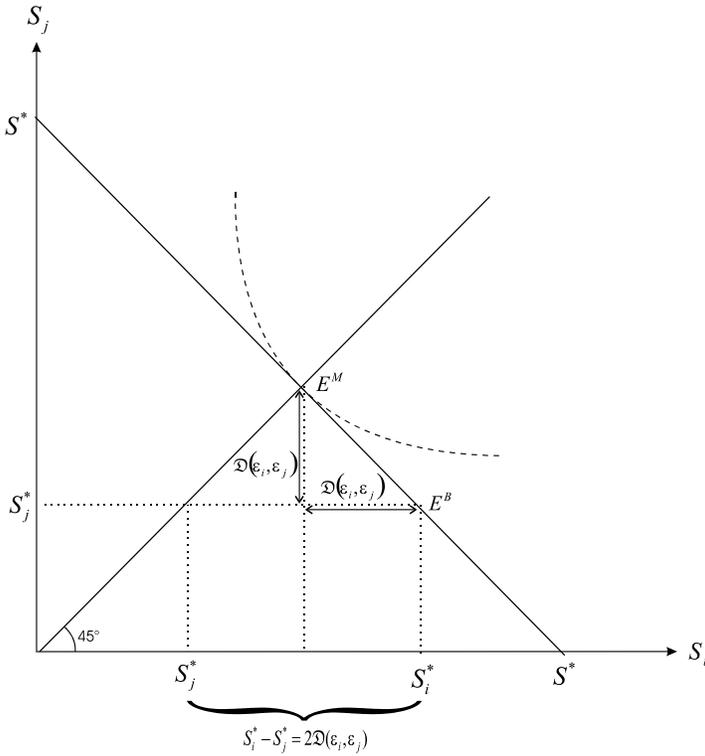


FIGURE 5

FRONTIER BETWEEN EFFICIENT AND NONEFFICIENT TRADES

produces the socially efficient quantities and when he receives  $m$  units of money is still smaller than  $S^*/2$ . The point  $E^M$  represents the surpluses of agents when efficient quantities are traded and when agents split the total surplus equally. Point  $E^B$  is the efficient allocation when agents produce efficient quantities and do not exchange money. To reach point  $E^M$  from  $E^B$ , agent  $i$  must transfer

$$\frac{S^*}{2} - S_j^* = \frac{S_i^* - S_j^*}{2}$$

in terms of utility to  $j$ . Thus, if  $(S_i^* - S_j^*)/2 \leq m\omega$ , the traders can attain  $E^M$ , because agent  $i$ 's real value of money holdings is sufficiently large. If, in contrast,  $(S_i^* - S_j^*)/2 > m\omega$ , the traders cannot attain allocation  $E^M$ , because agent  $i$  has not enough money to compensate  $j$ .

6. CONCLUSION

Most models that have captured the role of money as a medium of exchange have emphasized either its ability to increase the volume of trades or its ability to

solve an information problem. In this article we have focused on a complementary role by addressing the following question: Can money improve welfare in a double-coincidence-of-real-wants environment where in each bilateral meeting each agent is a consumer of the other agent's production?

Our answer is positive: If individuals have asymmetric preferences for each other's goods, money generates a strictly better allocation of resources. To study this question, we have considered a random-matching model with divisible money and divisible real commodities, where agents have the choice to finance current consumption with either money, real production, or both. In contrast to previous search models of money, in our environment money neither increases the frequency of trades nor solves an information problem. Rather, the welfare-improving role of money is based on its ability to allow agents to exploit higher gains from trade in asymmetric meetings than from trade attained in the same meetings in a barter economy.

We have found that if in a match the real value of money is sufficiently large, the traders produce, consume, and exchange socially efficient quantities, and they use money to split the total surplus of the match evenly. We have also shown that agents produce and consume the same (socially efficient) quantities and attain the same surpluses as if they had a device that allowed them to transfer utility perfectly, i.e., a device that allowed each agent to reduce her own payoff and increase the payoff of her partner in a match by the same amount. Thus, money is a *device* that transfers utility perfectly if its value is sufficiently high. The benefit of money arises because it allows agents to separate the decisions of how much to produce and exchange and how to split the resulting total surplus.

Although money is welfare-improving, inefficient monetary trades can occur if the degree of asymmetry in a match (measured by some distance) is larger than the real quantity of money, which is decreasing in the inflation rate. Consequently, inflation is costly in that it generates a misallocation of resources. A higher rate of expansion of the money supply increases the misallocation because it reduces the set of meetings where agents produce and exchange socially efficient quantities. When the gross growth rate of the money supply approaches some upper bound, the allocation of resources in the monetary economy converges to the (mis)allocation of the corresponding barter economy. In contrast, when the gross growth rate of the money supply approaches the discount factor, almost all trades are efficient, i.e., the Friedman rule holds.

There are several ways to extend our analysis. An interesting extension is to consider nondegenerate distributions of money holdings. With such a distribution, the marginal value of money will be different among individuals, and this could affect the role of money as a device to transfer utility. One could also consider a dual-currency or a two-countries and two-monies version of the model to study the determination of exchange rates and to compare the welfare properties of the model with those of the one-currency model. Finally, one could add some search externalities (by endogenizing the search intensity of traders, for instance) to see under which circumstances the Friedman rule still holds.

APPENDIX

A.1. *Welfare.* In the main text we compare the outcome of the decentralized economy with the outcome that a planner would choose in order to maximize welfare  $W$ . The social planner treats all households symmetrically and consequently maximizes the expected lifetime utility of the representative household  $W$  subject to the constraint that  $(q_\varepsilon^b, q_\varepsilon^s)$  must be equal to  $(q_{\varepsilon'}^s, q_{\varepsilon'}^b)$ :

$$(A.1) \quad \max_{q_\varepsilon^b, q_\varepsilon^s} W = \frac{\int_E [\varepsilon_i u(q_\varepsilon^b) - q_\varepsilon^s] d\mu(\varepsilon_i, \varepsilon_j)}{1 - \beta} \quad \text{s.t. } q_\varepsilon^b = q_{\varepsilon'}^s \quad \text{and} \quad q_\varepsilon^s = q_{\varepsilon'}^b$$

The problem can be reformulated as follows:

$$\max_{q_\varepsilon^b, q_\varepsilon^s} W = \frac{\int_{E'} [\varepsilon_i u(q_\varepsilon^b) - q_\varepsilon^s + \varepsilon_j u(q_\varepsilon^s) - q_\varepsilon^b] d\mu(\varepsilon_i, \varepsilon_j)}{1 - \beta}$$

where  $E' = \{(\varepsilon_i, \varepsilon_j) \in E \mid \varepsilon_i \geq \varepsilon_j\}$ . The first-order conditions are

$$\begin{aligned} \varepsilon_i u'(q_\varepsilon^b) &= 1 & \forall \varepsilon \in E \\ \varepsilon_j u'(q_\varepsilon^s) &= 1 & \forall \varepsilon \in E \end{aligned}$$

A.2. *Proofs.*

PROOF OF PROPOSITION 1. The demonstration proceeds in two parts.

*Part 1. Determination of the terms of trade.* Relabelling Equation (5) yields

$$(A.2) \quad \varepsilon_i u(q_{\varepsilon'}^s) - q_{\varepsilon'}^b = (1 - \delta \Delta) [\varepsilon_i u(q_\varepsilon^b) - q_\varepsilon^s]$$

For any  $\Delta$ , the terms of trade  $(q_\varepsilon^b, q_\varepsilon^s)$  and  $(q_{\varepsilon'}^b, q_{\varepsilon'}^s)$  are determined simultaneously and satisfy (4), (5), (A.2), and again (4) where  $(\varepsilon_i, \varepsilon_j)$  is replaced by  $(\varepsilon_j, \varepsilon_i)$ . In the following, we consider the solution when  $\Delta \rightarrow 0$ . First, Equations (4), (5), and (A.2) imply that  $\lim_{\Delta \rightarrow 0} q_\varepsilon^b = \lim_{\Delta \rightarrow 0} q_{\varepsilon'}^b$  and  $\lim_{\Delta \rightarrow 0} q_\varepsilon^s = \lim_{\Delta \rightarrow 0} q_{\varepsilon'}^s$ . Second, rearrange Equations (5) and (A.2) to get

$$\begin{aligned} [\varepsilon_j u(q_\varepsilon^s) - \varepsilon_j u(q_{\varepsilon'}^b)] - (q_\varepsilon^b - q_{\varepsilon'}^s) &= -\delta \Delta [\varepsilon_j u(q_{\varepsilon'}^b) - q_{\varepsilon'}^s] \\ [\varepsilon_i u(q_{\varepsilon'}^s) - \varepsilon_i u(q_\varepsilon^b)] - (q_{\varepsilon'}^b - q_\varepsilon^s) &= -\delta \Delta [\varepsilon_i u(q_\varepsilon^b) - q_\varepsilon^s] \end{aligned}$$

Divide the two last equations, use Equation (4), and take the limit  $\Delta \rightarrow 0$  to get (7).<sup>29</sup>

*Part 2. Existence and Uniqueness.* Equation (4) defines a negative relationship between  $q_\varepsilon^b$  and  $q_\varepsilon^s$ . Furthermore, according to (4),  $\lim_{q_\varepsilon^s \rightarrow 0} q_\varepsilon^b = +\infty$  and

<sup>29</sup> For more details and a related demonstration, see Osborne and Rubinstein (1990, Section 4.4) and Muthoo (1999).

$\lim_{q_\varepsilon^b \rightarrow +\infty} q_\varepsilon^b = 0$ . Equation (7) defines a positive relationship between  $q_\varepsilon^b$  and  $q_\varepsilon^s$ . Furthermore, according to (7),  $\lim_{q_\varepsilon^s \rightarrow 0} q_\varepsilon^b = 0$  and  $\lim_{q_\varepsilon^s \rightarrow +\infty} q_\varepsilon^b = +\infty$ . Hence, the terms of trade  $(q_\varepsilon^b, q_\varepsilon^s)$  are the unique solution of Equations (4) and (7). Accordingly, the barter equilibrium exists and is unique. ■

PROOF OF COROLLARY 1. Note, first, that  $\varepsilon_i > \varepsilon_j$  implies that  $q_\varepsilon^{b*} > q_\varepsilon^{s*}$ . Assume, next, that  $q_\varepsilon^b \geq q_\varepsilon^{b*}$ . This implies that  $\varepsilon_i u'(q_\varepsilon^b) \leq \varepsilon_i u'(q_\varepsilon^{b*}) = 1$ . Equation (4) implies  $q_\varepsilon^{s*} \geq q_\varepsilon^s$  and therefore,

$$(A.3) \quad \varepsilon_i u(q_\varepsilon^b) - q_\varepsilon^s \geq \varepsilon_i u(q_\varepsilon^{b*}) - q_\varepsilon^{s*} > \varepsilon_j u(q_\varepsilon^{s*}) - q_\varepsilon^{b*} \geq \varepsilon_j u(q_\varepsilon^s) - q_\varepsilon^b$$

According to (7),  $\varepsilon_j u(q_\varepsilon^s) - q_\varepsilon^b \geq \varepsilon_i u(q_\varepsilon^b) - q_\varepsilon^s$ , which contradicts (A.3). Thus, if  $\varepsilon_i > \varepsilon_j$ , we must have  $q_\varepsilon^b < q_\varepsilon^{b*}$  and, by (4),  $q_\varepsilon^s > q_\varepsilon^{s*}$ .

To see how the traders split the total surplus of the match, manipulate Equation (7) to get

$$(A.4) \quad \varepsilon_i u(q_\varepsilon^b) - q_\varepsilon^s = \frac{\varepsilon_i u'(q_\varepsilon^b)}{1 + \varepsilon_i u'(q_\varepsilon^b)} \{ \varepsilon_i u(q_\varepsilon^b) - q_\varepsilon^b + \varepsilon_j u(q_\varepsilon^s) - q_\varepsilon^s \}$$

Note, first, that  $q_\varepsilon^b < q_\varepsilon^{b*}$  implies  $\varepsilon_i u'(q_\varepsilon^b) > \varepsilon_i u'(q_\varepsilon^{b*}) = 1$ . This and Equation (A.4) imply that  $i$ 's fraction of the total surplus is  $\varepsilon_i u'(q_\varepsilon^b) / [1 + \varepsilon_i u'(q_\varepsilon^b)] > \frac{1}{2}$ .

To establish that  $q_\varepsilon^b < q_\varepsilon^s$ , rewrite (7) as follows:

$$\varepsilon_i u'(q_\varepsilon^b) [ \varepsilon_j u(q_\varepsilon^s) - q_\varepsilon^b ] = \varepsilon_i u(q_\varepsilon^b) - q_\varepsilon^s$$

Using (4) and after some manipulation, we obtain

$$\frac{u(q_\varepsilon^s)}{u'(q_\varepsilon^s)} + q_\varepsilon^s = \varepsilon_i u'(q_\varepsilon^b) \left[ \frac{u(q_\varepsilon^b)}{u'(q_\varepsilon^b)} + q_\varepsilon^b \right]$$

From the fact that  $\varepsilon_i u'(q_\varepsilon^b) > 1$  we deduce that

$$\frac{u(q_\varepsilon^s)}{u'(q_\varepsilon^s)} + q_\varepsilon^s > \frac{u(q_\varepsilon^b)}{u'(q_\varepsilon^b)} + q_\varepsilon^b$$

Consequently  $q_\varepsilon^b < q_\varepsilon^s$ . ■

PROOF OF LEMMA 1. By relabelling Equation (20), we get

$$(A.5) \quad \varepsilon_i u(q_\varepsilon^{s'}) - q_\varepsilon^{b'} + x_{\varepsilon'} \omega = (1 - \delta \Delta) [ \varepsilon_i u(q_\varepsilon^b) - q_\varepsilon^s - x_\varepsilon \omega ]$$

For any  $\Delta$  and a given  $(q_\varepsilon^{b'}, q_\varepsilon^{s'}, x_{\varepsilon'})$ , the offers  $(q_\varepsilon^b, q_\varepsilon^s, x_\varepsilon)$  and multipliers  $(\lambda_\varepsilon, \pi_\varepsilon)$  are determined by Equations (14)–(17) and (20). Moreover, for any  $\Delta$  and for a given  $(q_\varepsilon^b, q_\varepsilon^s, x_\varepsilon)$ , the offers  $(q_\varepsilon^{b'}, q_\varepsilon^{s'}, x_{\varepsilon'})$  and multipliers  $(\lambda_{\varepsilon'}, \pi_{\varepsilon'})$ , are determined by Equation (A.5) and by Equations (14)–(17) where  $(\varepsilon_i, \varepsilon_j)$  is replaced by  $(\varepsilon_j, \varepsilon_i)$ .

Note, first, that Equations (19), (20), and (A.5) imply that  $\lim_{\Delta \rightarrow 0} q_\varepsilon^b = \lim_{\Delta \rightarrow 0} q_{\varepsilon'}^s$ ,  $\lim_{\Delta \rightarrow 0} q_\varepsilon^s = \lim_{\Delta \rightarrow 0} q_{\varepsilon'}^b$ , and  $\lim_{\Delta \rightarrow 0} x_\varepsilon = \lim_{\Delta \rightarrow 0} -x_{\varepsilon'}$ . Moreover,  $\lambda_\varepsilon$  and  $\pi_{\varepsilon'}$  are either both positive or both equal to zero. To see this, suppose to the contrary that  $\lambda_\varepsilon > 0$  and  $\pi_{\varepsilon'} = 0$ . From (19),

$$\begin{aligned} \pi_{\varepsilon'} = 0 &\Rightarrow q_{\varepsilon'}^b \geq q_{\varepsilon'}^{b*} = q_\varepsilon^{s*} \text{ and } q_{\varepsilon'}^s \leq q_{\varepsilon'}^{s*} = q_\varepsilon^{b*} \\ \lambda_\varepsilon > 0 &\Rightarrow q_\varepsilon^b > q_\varepsilon^{b*} \text{ and } q_\varepsilon^s < q_\varepsilon^{s*} \end{aligned}$$

Furthermore, from (A.5),

$$\lim_{\Delta \rightarrow 0} \varepsilon_i u(q_{\varepsilon'}^s) - \varepsilon_i u(q_\varepsilon^b) + q_\varepsilon^s - q_{\varepsilon'}^b = \lim_{\Delta \rightarrow 0} [-(x_\varepsilon + x_{\varepsilon'})\omega]$$

The left-hand side is negative. Hence, the right-hand side must be negative too, which implies that  $x_{\varepsilon'} > m$ . This is impossible.

Note, next, that Equations (16) and (17) imply that  $\lambda_\varepsilon$  and  $\pi_\varepsilon$  cannot be positive at the same time. Consequently, there are two cases:

*Case (a) ( $\lambda_\varepsilon > 0$  and  $\pi_\varepsilon = 0$  or  $\lambda_\varepsilon = 0$  and  $\pi_\varepsilon > 0$ ).* Let us consider, first,  $\lambda_\varepsilon > 0$  and  $\pi_\varepsilon = 0$ . This implies that the constraint on money holdings of agent  $i$  is binding and  $x_\varepsilon = m$ . It also implies  $\lim_{\Delta \rightarrow 0} x_{\varepsilon'} = -m$ ,  $\lim_{\Delta \rightarrow 0} \lambda_{\varepsilon'} = 0$ , and  $\lim_{\Delta \rightarrow 0} \pi_{\varepsilon'} = \omega\lambda_\varepsilon/(\lambda_\varepsilon + \omega)$ . To see this, note that when  $\Delta$  approaches zero, (14) and (15) imply that

$$\frac{\omega}{\omega - \pi_{\varepsilon'}} = \frac{\lambda_\varepsilon + \omega}{\omega}$$

If  $\pi_\varepsilon = 0$ , the first-order conditions (14) and (15) equal

$$(A.6) \quad \varepsilon_i u'(q_\varepsilon^b) = \frac{\lambda_\varepsilon + \omega}{\omega}$$

$$(A.7) \quad \varepsilon_j u'(q_\varepsilon^s) = \frac{\omega}{\omega + \lambda_\varepsilon}$$

Equations (A.6) and (A.7) imply that if agent  $i$ 's constraint is binding,  $\varepsilon_i u'(q_\varepsilon^b) > 1 > \varepsilon_j u'(q_\varepsilon^s)$ , that is, the quantity  $q_\varepsilon^b$  produced and consumed is inefficiently low and the quantity  $q_\varepsilon^s$  produced and consumed is inefficiently large.

By rearranging (20) and (A.5), and by dividing these two equations, we get

$$(A.8) \quad \frac{[\varepsilon_j u(q_\varepsilon^s) - \varepsilon_j u(q_{\varepsilon'}^b)] - (q_\varepsilon^b - q_{\varepsilon'}^s) + (x_\varepsilon + x_{\varepsilon'})\omega}{[\varepsilon_i u(q_{\varepsilon'}^s) - \varepsilon_i u(q_\varepsilon^b)] - (q_{\varepsilon'}^b - q_\varepsilon^s) + (x_\varepsilon + x_{\varepsilon'})\omega} = \frac{\varepsilon_j u(q_{\varepsilon'}^b) - q_{\varepsilon'}^s - x_{\varepsilon'}\omega}{\varepsilon_i u(q_\varepsilon^b) - q_\varepsilon^s - x_\varepsilon\omega}$$

Take the limit as  $\Delta \rightarrow 0$  and use (19) to get Equation (21). If agent  $i$  is constrained by his money holdings, he receives more than one-half of the total surplus of the match. To see this, consider the following equation, which is derived from Equation (21):

$$(A.9) \quad \varepsilon_j u(q_\varepsilon^s) - q_\varepsilon^b + m\omega = \frac{\omega}{2\omega + \lambda_\varepsilon} [\varepsilon_i u(q_\varepsilon^b) - q_\varepsilon^b + \varepsilon_j u(q_\varepsilon^s) - q_\varepsilon^s]$$

According to (A.9), agent  $j$ 's fraction of the total surplus of the match is endogenous and depends on the ratio of the shadow price  $\lambda_\varepsilon$  to the marginal utility of money  $\omega$ . If  $\lambda_\varepsilon > 0$ , then  $\omega/(2\omega + \lambda_\varepsilon) < \frac{1}{2}$  and therefore  $\varepsilon_j u(q_\varepsilon^s) - q_\varepsilon^b + m\omega < \varepsilon_i u(q_\varepsilon^b) - q_\varepsilon^s - m\omega$ .

Next, consider  $\lambda_\varepsilon = 0$  and  $\pi_\varepsilon > 0$ . The constraint on money holdings of agent  $j$  is binding:  $x_\varepsilon = -m$ . This implies  $\lim_{\Delta \rightarrow 0} x_{\varepsilon'} = m$ ,  $\lim_{\Delta \rightarrow 0} \lambda_{\varepsilon'} = \omega\pi_\varepsilon/(\pi_\varepsilon + \omega)$ , and  $\lim_{\Delta \rightarrow 0} \pi_{\varepsilon'} = 0$ . Hence, we have  $(q_\varepsilon^b, q_\varepsilon^s) = (q_{\varepsilon'}^s, q_{\varepsilon'}^b)$ , where  $(q_{\varepsilon'}^b, q_{\varepsilon'}^s)$  satisfies (19) and (21).

Case (b) ( $\lambda_\varepsilon = 0$  and  $\pi_\varepsilon = 0$ ). This implies  $\lim_{\Delta \rightarrow 0} \lambda_{\varepsilon'} = \lim_{\Delta \rightarrow 0} \pi_{\varepsilon'} = 0$ . The first-order conditions (14) and (15) imply that  $q_\varepsilon^b = q_\varepsilon^{*b}$  and  $q_\varepsilon^s = q_\varepsilon^{*s}$ . Agents  $i$  and  $j$  trade efficient quantities. Equations (20) and (A.5) imply that if  $\Delta \rightarrow 0$ , we get an equation analogous to (A.8), and then (22): Traders split the total surplus of the match equally. ■

PROOF OF PROPOSITION 2. The demonstration proceeds in five steps.

1st step (Determination of the set  $I = I' \cup I''$ ; see Figure 6). Denote by  $I' \subseteq E$  the set of  $(\varepsilon_i, \varepsilon_j)$  such that  $\lambda_\varepsilon > 0$ . Note that  $\lambda_\varepsilon > 0$  implies that the constraint on agent  $i$ 's money holdings is binding and  $\varepsilon_i > \varepsilon_j$ . It also implies that agents  $i$  and  $j$

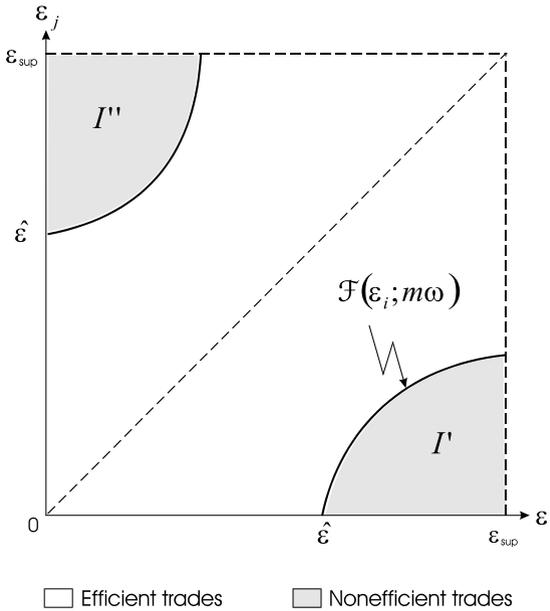


FIGURE 6  
SET OF INEFFICIENCIES

exchange inefficient quantities and they do not split the total surplus of the match equally. Hence,  $\lambda_\varepsilon > 0$  if (22) is satisfied for a value of  $x_\varepsilon > m$ . Thus,

$$(A.10) \quad 1 > \frac{\varepsilon_j u(q_\varepsilon^{s*}) - q_\varepsilon^{b*} + m\omega}{\varepsilon_i u(q_\varepsilon^{b*}) - q_\varepsilon^{s*} - m\omega}$$

From (6) we have  $q_\varepsilon^{b*} = u'^{-1}(1/\varepsilon_i)$  and  $q_\varepsilon^{s*} = u'^{-1}(1/\varepsilon_j)$ . Consequently, condition (A.10) can be rewritten as

$$(A.11) \quad \varphi(\varepsilon_i) - \varphi(\varepsilon_j) > 2m\omega$$

Thus, condition (A.11) is equivalent to  $\mathcal{D}(\varepsilon_i, \varepsilon_j) > m\omega$ .

Denote by  $\tilde{\varepsilon} = (\tilde{\varepsilon}_i, \tilde{\varepsilon}_j)$  the value of  $\varepsilon$  such that (A.11) is satisfied with equality:

$$(A.12) \quad \varphi(\tilde{\varepsilon}_i) - \varphi(\tilde{\varepsilon}_j) = 2m\omega$$

Equation (A.12) implicitly defines a function  $\mathcal{F}$  such that  $\tilde{\varepsilon}_j = \mathcal{F}(\tilde{\varepsilon}_i, m\omega)$ . Equation (A.12) implies that  $\mathcal{F}_1(\tilde{\varepsilon}_i, m\omega) = \partial\tilde{\varepsilon}_j/\partial\tilde{\varepsilon}_i > 0$  and that  $\mathcal{F}_2(\tilde{\varepsilon}_i, m\omega) = \partial\tilde{\varepsilon}_j/\partial m\omega < 0$ . Furthermore, for all  $m\omega > 0$ , we have  $\mathcal{F}(\varepsilon_{\text{sup}}; m\omega) < \varepsilon_{\text{sup}}$ , and there exists  $\hat{\varepsilon}(m\omega) > 0$  such that  $\mathcal{F}(\hat{\varepsilon}; m\omega) = 0$  (see Figure 5). Consequently, the set  $I'$  is defined as follows:

$$\begin{aligned} I' &= \{(\varepsilon_i, \varepsilon_j) \in E \mid \varphi(\varepsilon_i) - \varphi(\varepsilon_j) > 2m\omega\} \\ &= \{(\varepsilon_i, \varepsilon_j) \in E \mid \varepsilon_j < \mathcal{F}(\varepsilon_i)\} \end{aligned}$$

By symmetry, we can define the set  $I''$  of  $\varepsilon$  such that  $\pi_\varepsilon > 0$ :

$$I'' = \{(\varepsilon_i, \varepsilon_j) \in E \mid (\varepsilon_j, \varepsilon_i) \in I'\}$$

Accordingly,  $I = I' \cup I''$ ; see Figure 5.

*2nd step (The measure of inefficient trades).* In order to measure the set of inefficient trades, it is useful to introduce the indicator function  $\mathbf{1}_A(x)$  that is equal to 1 if  $x \in A$  and 0 otherwise. Furthermore, because of the symmetry of the sets  $I'$  and  $I''$  we have  $\mu(I) = 2\mu(I')$ .

Let us calculate  $\mu(I')$ . By definition,

$$\begin{aligned} \mu(I') &= \int \mathbf{1}_{[0, \mathcal{F}(\varepsilon_i; m\omega)]}(\varepsilon_j) d\mu(\varepsilon_i, \varepsilon_j) \\ &= \int_{[0, \varepsilon_{\text{sup}}]} \left( \int_{[0, \varepsilon_{\text{sup}}]} \mathbf{1}_{[0, \mathcal{F}(\varepsilon_i; m\omega)]}(\varepsilon_j) dF(\varepsilon_j) \right) dF(\varepsilon_i) \end{aligned}$$

This integral indicates that we measure the set of points such that  $\varepsilon_j < \mathcal{F}(\varepsilon_i)$  with the measure  $\mu$ . The integral  $\int_{[0, \varepsilon_{\text{sup}}]} \mathbf{1}_{[0, \mathcal{F}(\varepsilon_i; m\omega)]}(\varepsilon_j) dF(\varepsilon_j) = \int_0^{\mathcal{F}(\varepsilon_i; m\omega)} dF(\varepsilon_j)$  is equal to  $F[\mathcal{F}(\varepsilon_i; m\omega)]$  for all  $\varepsilon_i > \hat{\varepsilon}$  and zero otherwise. Hence,

$$\mu(I') = \int_{[\hat{\varepsilon}, \varepsilon_{\text{sup}}]} F[\mathcal{F}(\varepsilon_i; m\omega)] dF(\varepsilon_i)$$

By differentiating  $\mu(I')$  with respect to  $m\omega$  we get

$$\frac{\partial \mu(I')}{\partial m\omega} = \int_{[\hat{\varepsilon}, \varepsilon_{\text{sup}}]} f[\mathcal{F}(\varepsilon_i; m\omega)] \times \mathcal{F}_2(\varepsilon_i; m\omega) dF(\varepsilon_i) - F[\mathcal{F}(\hat{\varepsilon}; m\omega)] \frac{\partial \hat{\varepsilon}}{\partial m\omega}$$

Because  $F[\mathcal{F}(\hat{\varepsilon}; m\omega)] = 0$ , we have

$$\frac{\partial \mu(I')}{\partial m\omega} = \int_{[\hat{\varepsilon}, \varepsilon_{\text{sup}}]} f[\mathcal{F}(\varepsilon_i; m\omega)] \times \mathcal{F}_2(\varepsilon_i; m\omega) dF(\varepsilon_i) < 0$$

Consequently, the set of inefficiencies  $\mu(I) = 2\mu(I')$  is decreasing in  $m\omega$ .

*3rd step (Determination of  $m\omega$ ).* The quantity  $m\omega$  is determined by the envelope condition (18), which can be rewritten as follows:

$$\frac{\omega_{-1}}{\beta} = \int_{I'} \lambda_{\varepsilon} d\mu(\varepsilon_i, \varepsilon_j) + \omega$$

By dividing by  $\omega/\beta$ , we get

$$(A.13) \quad \frac{\omega_{-1}}{\omega} = \beta \left\{ \int_{I'} \frac{\lambda_{\varepsilon}}{\omega} d\mu(\varepsilon_i, \varepsilon_j) + 1 \right\}$$

The rate of growth of money supply is  $\gamma - 1$ . Thus,

$$\frac{m}{m_{-1}} = \gamma$$

In the steady state,  $q_{\varepsilon}^b = q_{\varepsilon, -1}^b$  and  $q_{\varepsilon}^s = q_{\varepsilon, -1}^s$  for all  $\varepsilon$ . In the following we demonstrate that  $q_{\varepsilon}^b$  and  $q_{\varepsilon}^s$  depend on  $m\omega$ , which implies that  $m\omega$  is constant in the steady state, and therefore

$$\frac{\omega_{-1}}{\omega} = \gamma$$

Hence, Equation (A.13) gives

$$(A.14) \quad \gamma = \beta \left\{ \int_{I'} \frac{\lambda_{\varepsilon}}{\omega} d\mu(\varepsilon_i, \varepsilon_j) + 1 \right\}$$

Equations (A.6) and (A.7) imply that  $q_{\varepsilon}^b$  is decreasing in  $\lambda_{\varepsilon}/\omega$  and  $q_{\varepsilon}^s$  is increasing in  $\lambda_{\varepsilon}/\omega$ . If  $\lambda_{\varepsilon} > 0$ , then  $\varepsilon_i u'(q_{\varepsilon}^b) > 1$  and  $\varepsilon_j u'(q_{\varepsilon}^s) < 1$ . Hence, the left-hand side of

(A.9) is increasing in  $\lambda_\varepsilon/\omega$ , and the right-hand side of (A.9) is decreasing in  $\lambda_\varepsilon/\omega$ . Consequently, Equation (A.9) determines a unique value for  $\lambda_\varepsilon/\omega$ . Moreover,

$$\frac{\partial \frac{\lambda_\varepsilon}{\omega}}{\partial m\omega} < 0, \quad \frac{\partial q_\varepsilon^b}{\partial m\omega} > 0, \quad \frac{\partial q_\varepsilon^s}{\partial m\omega} < 0$$

Equation (A.9) implicitly defines a function  $\psi(\varepsilon; m\omega) = \lambda_\varepsilon/\omega$  for all  $\varepsilon \in I'$  with  $\partial\psi/\partial m\omega < 0$ . Consequently, Equation (A.14) can be rewritten as

$$(A.15) \quad \gamma = \beta \left\{ \int_{I'} \psi(\varepsilon; m\omega) d\mu(\varepsilon_i, \varepsilon_j) + 1 \right\}$$

From (A.15),  $m\omega$  is entirely characterized by the following equation:

$$(A.16) \quad \Psi(m\omega) = \frac{\gamma}{\beta} - 1$$

where  $\Psi(m\omega) = \int_{I'} \psi(\varepsilon; m\omega) d\mu(\varepsilon_i, \varepsilon_j)$ . By differentiating (A.16) with respect to  $m\omega$ , we find  $\Psi' < 0$  as long as  $\mu(I') > 0$ . This implies that if there is a  $m\omega > 0$  satisfying (A.16) such that  $\mu(I') > 0$ , then it is unique.

*4th step (Existence of a unique monetary equilibrium).* For existence and uniqueness we need to show that there is unique  $x > 0$  satisfying  $\Psi(x) = \frac{\gamma}{\beta} - 1$ . To do this we study the properties of  $\Psi(x)$ . Let us first derive  $\Psi(0)$ . If  $m\omega = 0$ , Equations (14) and (A.9) are equivalent to (A.4), and we have  $q_\varepsilon^s = \hat{q}_\varepsilon^s$  and  $q_\varepsilon^b = \hat{q}_\varepsilon^b$ , where  $\hat{q}_\varepsilon^s$  and  $\hat{q}_\varepsilon^b$  are the quantities exchanged in the barter economy. Furthermore,

$$\frac{\lambda_\varepsilon}{\omega} = \varepsilon_i u'(\hat{q}_\varepsilon^b) - 1$$

Then

$$\Psi(0) = \int_E \max(\varepsilon_i u'(\hat{q}_\varepsilon^b) - 1, 0) d\mu(\varepsilon_i, \varepsilon_j) > 0$$

Second, there is a critical value  $\bar{x} > 0$  such that for all  $x \geq \bar{x}$ ,  $\Psi(x) = 0$  and for all  $x < \bar{x}$ ,  $\Psi(x) > 0$ . To see this, note that Equation (A.10) implies that  $I' = \emptyset$  when  $m\omega$  is large enough. Finally, from step 3,  $\Psi'(x) < 0$  if  $x < \bar{x}$ .

We conclude that Equation (A.16) determines a unique positive value of  $m\omega$  if  $\gamma \in (\beta, \bar{\gamma})$ , where

$$\bar{\gamma} = \beta \int_E \max(\varepsilon_i u'(\hat{q}_\varepsilon^b), 1) d\mu(\varepsilon_i, \varepsilon_j)$$

*5th step (The limit case  $\gamma \rightarrow \beta$ ).* From (A.16),  $\Psi(m\omega) \rightarrow 0$ , that is,

$$\lim_{\gamma \rightarrow \beta} \int_{I'} \frac{\lambda_\varepsilon}{\omega} d\mu(\varepsilon_i, \varepsilon_j) = 0$$

This implies that  $\lim_{\gamma \rightarrow \beta} \mu(I') = 0$ : in the limit traders exchange efficient quantities in all matches. ■

A.3. *Derivation of the Pareto Frontier of the Bargaining Set.* The surpluses of agents  $i$  and  $j$  are

$$(A.17) \quad S_i = \varepsilon_i u(q_\varepsilon^b) - q_\varepsilon^s - x_\varepsilon \omega$$

$$(A.18) \quad S_j = \varepsilon_j u(q_\varepsilon^s) - q_\varepsilon^b + x_\varepsilon \omega$$

The Pareto frontier of the bargaining set is the set of pairs  $(S_i, S_j)$  such that it is not possible to increase  $S_i$  without decreasing  $S_j$ . Formally, the Pareto frontier is determined by the following program:

$$(A.19) \quad \begin{aligned} \max_{q_\varepsilon^b, q_\varepsilon^s, x_\varepsilon} \quad & S_i = \varepsilon_i u(q_\varepsilon^b) - q_\varepsilon^s - x_\varepsilon \omega \\ \text{s.t.} \quad & \varepsilon_j u(q_\varepsilon^s) - q_\varepsilon^b + x_\varepsilon \omega \geq S_j \end{aligned}$$

$$(A.20) \quad -m \leq x_\varepsilon$$

$$(A.21) \quad x_\varepsilon \leq m$$

Denote by  $\mu_\varepsilon$ ,  $\lambda_\varepsilon$ , and  $\pi_\varepsilon$  the multipliers associated with the inequalities (A.19), (A.20), and (A.21), respectively. In a barter economy  $\omega = 0$ , whereas in a monetary economy  $\omega > 0$ .

- The Pareto frontier in the barter economy  
The Lagrangian of this program is

$$\mathcal{L} = \varepsilon_i u(q_\varepsilon^b) - q_\varepsilon^s + \mu_\varepsilon [\varepsilon_j u(q_\varepsilon^s) - q_\varepsilon^b - S_j]$$

The first-order conditions with respect to  $q_\varepsilon^b$  and  $q_\varepsilon^s$  are

$$(A.22) \quad \varepsilon_i u'(q_\varepsilon^b) = \mu_\varepsilon$$

$$(A.23) \quad \varepsilon_j u'(q_\varepsilon^s) = \frac{1}{\mu_\varepsilon}$$

From (A.22) and (A.23) we have

$$(A.24) \quad \varepsilon_i \varepsilon_j u'(q_\varepsilon^b) u'(q_\varepsilon^s) = 1$$

Equations (A.17), (A.18), and (A.24) define a negative relationship between  $S_i$  and  $S_j$ , which is the Pareto frontier denoted by  $\mathcal{P}^B$ . One can show that the slope of  $\mathcal{P}^B$  is negative, i.e.,  $dS_j/dS_i = -1/\varepsilon_i u'(q_\varepsilon^b) < 0$ . Moreover, because  $S_i$  is an increasing function of  $q_\varepsilon^b$ ,  $\mathcal{P}^B$  is concave, i.e.,  $d^2 S_j/dS_i^2 < 0$ .

- The Pareto frontier in the monetary economy

The Pareto frontier in the monetary economy is denoted by  $\mathcal{P}^M$ . The Lagrangian has the following expression:

$$\begin{aligned} \mathcal{L} = \quad & \varepsilon_i u(q_\varepsilon^b) - q_\varepsilon^s - x_\varepsilon \omega + \mu_\varepsilon [\varepsilon_j u(q_\varepsilon^s) - q_\varepsilon^b + x_\varepsilon \omega - S_j] \\ & - \lambda_\varepsilon (x_\varepsilon - m) + \pi_\varepsilon (x_\varepsilon + m) \end{aligned}$$

The first-order conditions with respect to  $q_\varepsilon^b$ ,  $q_\varepsilon^s$  and  $x_\varepsilon$  are

$$(A.25) \quad \varepsilon_i u'(q_\varepsilon^b) = \mu_\varepsilon$$

$$(A.26) \quad \varepsilon_j u'(q_\varepsilon^s) = \frac{1}{\mu_\varepsilon}$$

$$(A.27) \quad \omega(\mu_\varepsilon - 1) = \lambda_\varepsilon - \pi_\varepsilon$$

Again, from (A.25) and (A.26) we find that

$$(A.28) \quad \varepsilon_i \varepsilon_j u'(q_\varepsilon^b) u'(q_\varepsilon^s) = 1$$

There are three cases to distinguish: (i)  $\lambda_\varepsilon = \pi_\varepsilon = 0$ , (ii)  $\lambda_\varepsilon > 0$  and  $\pi_\varepsilon = 0$ , and (iii)  $\lambda_\varepsilon = 0$  and  $\pi_\varepsilon > 0$ .

*Case (i)* ( $\lambda_\varepsilon = \pi_\varepsilon = 0$ ). From (A.27) we have  $\mu_\varepsilon = 1$ . From (A.25) and (A.26)  $q_\varepsilon^b = q_\varepsilon^{b*}$  and  $q_\varepsilon^s = q_\varepsilon^{s*}$ . As a consequence, according to (A.17) and (A.18),

$$S_i = S_i^* - x_\varepsilon \omega, \quad S_j = S_j^* + x_\varepsilon \omega \quad \text{and} \quad S_i + S_j = S^*$$

If the constraints on money holdings are not binding, the points of the Pareto frontier lie on the line  $S^* S^*$ . Moreover, the constraint  $-m \leq x_\varepsilon \leq m$  can be rewritten as follows:

$$-m\omega + S_i^* \leq S_i \leq m\omega + S_i^*$$

*Case (ii)* ( $\lambda_\varepsilon > 0$  and  $\pi_\varepsilon = 0$ ). We have  $x_\varepsilon = m$  and, from (A.27),  $\mu_\varepsilon = 1 + \frac{\lambda_\varepsilon}{\omega}$ . Hence, one can see that (A.25) and (A.26) are exactly the first-order conditions of the program of the household. Furthermore, (A.17) and (A.18) yield

$$\begin{aligned} S_i + m\omega &= \varepsilon_i u(q_\varepsilon^b) - q_\varepsilon^s \\ S_j - m\omega &= \varepsilon_j u(q_\varepsilon^s) - q_\varepsilon^b \end{aligned}$$

Given that  $(q_\varepsilon^b, q_\varepsilon^s)$  satisfies (A.24), we deduce that  $(S_i + m\omega, S_j - m\omega) \in \mathcal{P}^B$ . Furthermore, the condition  $\lambda_\varepsilon > 0$  implies  $\mu_\varepsilon > 1$  and  $q_\varepsilon^b < q_\varepsilon^{b*}$  and  $q_\varepsilon^s > q_\varepsilon^{s*}$ . As a consequence,  $S_i + m\omega < S_i^*$ .

*Case (iii)* ( $\lambda_\varepsilon = 0$  and  $\pi_\varepsilon > 0$ ). The reasoning is similar to case (ii). We have  $x_\varepsilon = -m$  and  $(S_i - m\omega, S_j + m\omega) \in \mathcal{P}^B$ . This occurs when  $S_i - m\omega > S_i^*$ .

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